Handbook of International Research in Mathematics Education

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Contents

25 The Use of Digital Technology in Mathematical Practices

Reconciling Traditional and Emerging Approaches

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Cinvestav-IPN, México

INTRODUCTION

The pervasive presence of digital technologies in society, and their transformative powers in this second decade of the 21st century, remind us that we have entered new times full of challenges for educational systems. This is globally true even if one recognizes different levels of economic development of countries around the world. Significant and increasing availability of digital technologies has opened windows for people to explore new social spaces of participation and eventually redefine their own identity. In Mexico, there are about 99 million mobile phones, which means 86 phones per each 100 inhabitants. During a trip in the subway, in Mexico City, one can observe how people become isolated whilst using their phones (basic cell phones and even smartphones) for messaging and playing games. Isolation, however, is apparent as the phones *mediates* their presence in another place. Identity involves presence in social space, but now, social space is extended into a realm of virtual reality, on a permanent basis. The phone is the key to enter and participate in this enlarged infrastructure of society. Friedman (2007) uses the notion of *flatness* to explain and argue that, with the use of technology resources, more people are involved or can directly collaborate in addressing and discussing societal concerns than ever before. He writes: "what the flattening of the world means is that we are now connecting all the knowledge centers on the planet together into a single global network . . . in an amazing era of prosperity, innovation, and collaboration, by companies, communities, and individuals" (p. 8). Digital technologies have transformed and are transforming human relations and human cognitive powers. Friedman's book, among others, provides ample evidence of this fact. At the moment, the use of cognitive technologies (Pea, 1985) could be seen as *amplifiers* of human cognition. For instance, the use of a handheld calculator with Computer Algebra Systems (CAS) can help us solve problems that involve finding the roots of a given polynomial. This is something we could do without the handheld device, but it is faster and convenient to rely on this recourse. Like a magnifying glass, a cognitive technology can improve an ability we already possess. People usually develop this cognitive affordance when they begin representing and exploring tasks through this technology. However, in the long run, this is not quite the only proper role. Like a Trojan horse, a cognitive technology begins working stealthily in our mind and after a while it becomes part of our cognitive resources. This is the case with the technology of writing, for instance. As Donald (2001, p. 302) has explained, literacy skills transform the functional architecture of the brain and have a profound impact on *how literate people perform their cognitive work*. The complex neural components of a literate vocabulary, Donald explains, have to be hammered by years of schooling to rewire the functional organization of our thinking. Similarly, the decimal system (Kaput & Schorr, 2008, p. 212) first enlarged access to computation and eventually paved the way to the Modern Age.

Today, we cannot imagine the world without these technologies. They have become part of our infrastructure—obviously much more than mere amplifiers. That is, they have become essential tools that everyone learns and uses to sustain individual and social activities. They have rewired, as Donald (2001) wrote, the functional organization of individual brains and, at the same time, have become coextensive with our culture. It is the omnipresence of technologies in society that eventually endow them with ***invisibility***: they blend into society, as people are increasingly accustomed to their effects.

Thus, technologies eventually become natural and transparent in the social world, the truly human ecology.

We need to develop the critical capabilities to translate scientific and technological developments into our realities, more importantly, into our ***educational*** realities. Scientific knowledge undergoes tangible transformations before entering into the classroom. One comprehensive framework to guide and understand this translation of scientific knowledge into the knowledge taught in schools is provided by the theory of ***didactic transposition*** (Chevallard, 1985). Put simply, didactic transposition includes ways to reorganize knowledge so that the new resulting version is available as educational material.

This transposition creates a tension between social expectations, on one side, and what an educational system can deliver and offer to learners on the other. It is important to recognize that a school culture always leaves significant marks on students and teachers' values. As Artigue (2002) has aptly expressed, "these [culture] values were established, through history, in environments poor in technology, and they have only slowly come to terms with the evolution of mathematical practice linked to technological evolution" (p. 245). However, there is a fact that must be singled out: The emergent knowledge produced through the digital media is different from the knowledge emerging from a paper-and-pencil medium, because the mediating artifact is not epistemologically neutral. That is, the nature of the knowledge is inextricably linked to the mediating artifact (Moreno-Armella & Hegedus, 2009, p. 501). We will have an opportunity to discuss this issue broadly, later in this chapter.

It is important to recognize the existence of a natural tension between the past and the future, but it is also possible to resolve it if we realize that the prudent face looks into the past, and the innovative face looks into the future.

Today, our students are increasingly digital natives—and as teachers, we are digital immigrants (Prensky, 2010). Yet, even if we speak (digital) technology with an accent, we need to blend past technologies with the new ones.

In this context, school culture requires a gradual but permanent reorientation of its practices, and of its cognitive and epistemological assumptions, for students to gain access to powerful mathematical ideas. In our view, the classroom should be conceived of as the central nervous system of the educational process. However, that classrooom, as well as the educational system in toto, are open systems and consequently are under the multidimensional influence of its social and cultural environments.

Today, we have new ways to represent and communicate our experiences, in particular, to communicate the knowledge we have acquired. For example, communication technologies facilitate not only direct interaction among research communities, but also the sharing of experiences and results.

Bottino, Artigue, & Noss (2009) present a collaboration project that involves several European research teams discussing goals and ways to frame technology-enhanced learning from different theoretical traditions. But all this is not just about knowledge: it is centrally about ***knowing***. As Schmidt and Cohen (2013) pointed out, a computer, in 2025, will be 64 times faster than it is in 2013. This is a huge increase in computational power that should help individuals reorganize their ways of thinking, including their problem solving approaches. We cannot foresee, today, what this would imply for society in general and for education in particular in the next decades.

Mathematics is part of our culture and lives; it is embedded in every digital artifact, phone, computer, eBook, and so on. Eventually, we are compelled to ask: What is the *new* role of mathematics in contemporary societies increasingly saturated by the use of digital artifacts? How can we use available technologies (including smartphones) to foster students' development of sense-making actitivies and reasoning? Thus, we are forced to understand the strategies that teachers follow to appropriate the digital artifacts at their reach. For instance they can use ***conveyance technologies*** or ***mathematical action technologies*** (Dick & Hollebrands, 2011). On one hand, the former allow the teacher to present or communicate mathematical ideas in the classroom. Even if these technologies are not mathematics specific (Microsoft PowerPoint, LCD projectors, for instance) they are important for integrating the classroom around the discussion of someone's point of view with respect to a mathematical idea. Mathematical action technologies (Dick & Hollebrands, 2011), on the other hand, are used to activate and improve exploration, conjecture formulation, argumentation, and in general, mathematical ways of thinking.

A FOCUS ON MATHEMATICAL TASKS

In the last 10 years we have consistently been involved in several national research projects that aim to analyze and discuss the extent to which the use of digital technologies provide teachers and learners with new avenues to grasp and develop mathematical knowledge (Moreno-Armella & Santos-Trigo, 2008). During the development of those projects we have addressed themes related to teachers' involvement in problem- solving activities that enhance the use of several digital tools, curriculum reforms, and ways to design and implement mathematical tasks in actual learning scenarios (Santos-Trigo & Camacho-Machín, 2009). Our research approach includes working directly with teachers at public institutions through seminars and workshops. There, teachers have an opportunity to identify and discuss international developments around the use of digital technologies such as those published in handbooks and research journals, and ways to frame them in their actual teaching practices. Indeed, several of the tasks used in this chapter to illustrate ways of reasoning that emerge when learners think of and approach the tasks through the tools' affordances came from those projects. That is, tasks play an important role not only in fostering learners' construction of mathematical knowledge, but also in documenting students' ways of reasoning associated with the use of digital technology.

It is a truism that education necessitates the permanent and sustainable transformation of teachers. In Mexico, a country with a population of 112 million, there are 34.8 million students and about 1.8 million teachers. The student population between 15–18 years old represents 12.5% and this is the sector that will grow faster in coming years. There are 286,000 teachers for this sector and 328,000 university teachers. At the university level there are 3.2 million students, which represents 9.1% of the global student population. These updated figures come from the Ministry of Education (SEP) and offer a partial view of the social realities that constitute the environment where teachers work and will develop their professional lives. The tension between local traditions and global transformations, or national and international innovation, is permanent. We need, as educational researchers, to transform ideas that we think are important and spread them along the permanent professional development of teachers. They are our closest colleagues.

GUIDE AND BEING GUIDED BY AN ARTIFACT

Imagine the early encounter of a student with a violin. After the first sessions, the student comes home with pain in her shoulders, neck, and hands. Perhaps the violin is out of tune, but in certain ways the violin is presenting some resistance to the student's efforts to conquer it. These initial obstacles and shortcomings are some of the constraints the artifact imposes on a would-be-violinist. One could say that the student's learning is *being guided* by the structure of the violin. Imagine, now, the same student some 15 years later, as a brilliant professional violinist, in the middle of a concert. No pain in her hands, neck, or shoulders. The music flows smoothly with her performance and now, the violin is *invisible*, meaning that the violinist overcame all those early shortcomings years ago: today, the musician *guides* the violin to express her music and her artistic sensibilities. It is as if the violinist was able to create a distortion of reality *mediated* by the violin (almost) an organic part of her skills. In our view, artifacts and activity influence each other, flow through each other, and even more: the music, the violin, and the artist are coextensive, coterminous. Thus, the subjects' activities are mediated by the use of the artifact, which also influences the subjects' actions.

The long bidirectional process through which the music student *internalizes* the violin, overcoming all its resistances, takes place within a cultural medium. It is within a culture that the musician finds the motives and the concomitant aesthetic values that fill her efforts with meaning.

This short narrative aims to describe the nature of the relationship between a person and an artifact that she/he wants to use for accomplishing a task. There is a deep level of complexity—technical, cultural, and cognitive—implicit in this narrative. We shall try to reveal a significant piece of this complexity in the following pages when we reflect on learning and teaching mathematics with the mediation of digital technologies.

Working with teachers in our graduate program has been an invaluable opportunity to learn how they deal with digital technologies like Geogebra, installed in computers, for instance. The teachers are motivated because they will use this technology when they return to their classrooms. There is a professional commitment as well as an increasing social pressure to gain fluency with these artifacts—we find cell phones in the subway as well as in the schools.

In the first decades of the present century there have been serious efforts to *problematize* the presence of CAS and Dynamic Geometry Software (DGS) in the classroom. Besides having been installed in material artifacts (calculators, computers, smartphones, and so on), CAS and DGS are semiotic artifacts because they *mediate* semiotic tasks when we are, for instance, trying to coordinate several symbolic registers of a mathematical object. We can illustrate this with the analysis of area variation of a family of rectangles with fixed perimeter through a discrete approach (a table), a graphic generated via loci of points, or in terms of an algebraic model.

Teachers need to understand the workings of these artifacts, and their syntactical rules, in order to use them meaningfully as *mediators* of mathematical knowledge. For this to happen, there must be a melody to be played, that is, teachers need a mathematical task. The task is an incentive for teachers to figure out how to integrate in meaningful ways the symbolic artifact to their own intellectual resources in order to solve that task. If a person succeeds in integrating the artifact to his/her cognitive resources to solve a task, then, Verillon and Rabardel (1995) explain that the artifact has become an *instrument*. For instance, when we compute the multiplication of two large numbers our cognitive activity is mediated by the positional system we use to represent numbers. We find it very natural to proceed as we usually do. The positional system in base 10 is more than a cultural artifact: it has become an instrument of our reason to deal with numbers. This is in sheer contrast to computing with numbers written in base, say, seven. In this case we have a cultural artifact that most people have not transformed into an instrument to think in numbers and solve problems.

Let us introduce another example: An architect begins using specific software to design his buildings. Taking profit from the plasticity of the visual images that the software provides, the architect can imagine a new plan, a new design. Gradually he will begin thinking of his design *with* and *through* the software. The architect will incorporate the software as part of his thinking and one day, the software will have *disappeared* as such. Now it is *coextensive* with the architect's thinking whilst solving design tasks. The software has become an instrument and the design activities are ***instrumented*** activities.

A SYMBOLIC MODE OF EXISTENCE

The drawing of a chair is a representation of the chair, but it is not the actual chair: we cannot sit down on the drawing. We are reminded of the famous René Magritte's painting *The Treachery of Images*: there is the image of a pipe and, below the image, the words: *ceci n'est pas une pipe* (this is not a pipe). Imagine Magritte had drawn a triangle instead of the pipe: this is not a triangle. Knowing that he could smoke with the real pipe ***represented*** on the canvas, we can ask: Can he bring to the fore a ***real*** triangle? Obviously, the answer is no. However, if we have a representation of the triangle, what is that "other thing" that is being represented on the canvas? Of course it is not a concrete, material thing, like Magritte's pipe.

We have to look for a different answer. Writing has the power to crystallize an idea and give it a tangible level of objectivity. This is what happens with mathematical notation systems, i.e., symbolic representations. But how does it happen in mathematics? For instance, we have the experience of periodicity (day-night-day again) and we create a periodic function—that is, a conceptual entity—to deal with the diversity of these phenomena. At its early stages, mathematical concepts are born from human perceptions or conceived from activities like building a round object. Later, the symbolic representation that captures our original perception or experience establishes links with other concepts and eventually becomes the initial steps for even further elaboration of mathematical concepts. This is similar to what we find in a dictionary: one looks for the meaning of a word and the dictionary provides a description with other words. Words do not live a full life if they are isolated; they need to live in networks. This happens to mathematical concepts as well. For instance, students can easily search for online sources to explore a concept's meaning and applications, and they can even ask and discuss their peers' views and comprehension of those concepts.

With teachers we have used this example: It is known that the number of atoms in the universe is of the order 10^{85}. Then, what is the meaning of the ***number*** 10^{400} ? Apparently, there are numbers that have no referent but they still are numbers: they are the result, for instance, of arithmetic operations. This problem, simple as it appears to be, opens the door to interesting (epistemological) discussions with teachers. Reference is found not only in the material world, indeed, reference is found in the world of human actions that extend the material world.

So far, we have been trying to depict the conceptual path walked through with teachers. We have been reminded of the importance to invite the teachers to communal reflections and discussions about conceptual difficulties that may be lurking ahead when a student tries to appropriate a piece of mathematics at school.

Saying that a mathematical entity is a cultural object crystallized by symbolic means, from human activity, needs detailed unpacking, especially if one is thinking of education. René Thom, in his plenary lecture at ICME 2, 1972 said: "The real problem that confronts mathematics teaching is not that of rigor but the problem of the development of meaning, of the existence of mathematical objects" (Thom, 1972, p. 202)

One of us remembers a geometry class where students were asked to prove that in a triangle the length of each side is less than the sum of the lengths of the other two sides. A student came to the board with a cord to measure the sum of two sides and verify that the resulting piece of cord was longer than the third side. We had previously discussed that a straight line was *like* a tight cord. The discussion that students engaged (including a discussion about the triangle Earth-Sun-Moon) to prove the theorem, taught the teacher how difficult it was for students to understand that a mathematical object was a conceptual entity and that the only way to access it is through a symbolic representation (Duval, 2006). Even more, one representation is not enough to exhaust all the features of a mathematical entity. In that sense, the mathematical entity is always under construction. It is the lack of ostensive referrals to mathematical entities that generates this sense of elusiveness that students feel, quite often, whilst dealing with mathematical problems. However, it is a matter of levels of abstraction and generality that we have to deal with. Students need to feel they are dealing with something that has a palpable existence (Lakoff & Nuñez, 2000). Indeed, there is evidence that the process of learning a concept is facilitated when the student has the opportunity to work with a rich diversity of symbolic representations of that concept.

To approach these and other delicate matters, we turn to digital media.

VIGNETTES AND EXEMPLARS: GEOMETRY AND CALCULUS.

Like the two faces of Janus that look to the past and to the future, education looks to tradition and innovation. We cannot forget that today's curriculum has deep roots in the ways mathematics has been conceived traditionally with paper and pencil—and we cannot forget, either, the importance of the available digital armamentarium, resources that teachers can incorporate into their teaching practices. But even if digital technologies are not fully integrated within the school mathematical universe, they will gradually erode and transform the mathematical ways of thinking embedded in the traditional system. In Mexico, a government proposal to provide online resources to all elementary schools became important to discuss ways to transform textbooks into interactive materials where students explore mathematical ideas from diverse perspectives.

To consider this dilemma, the tension between tradition and innovation, we explore digital representations of mathematical entities. Doing so will reveal properties of these entities that lie hidden or opaque, to begin with. Our goal is to develop a ***transitional way of thinking*** more in agreement with the requirements of education today. For instance, simple mathematics objects such as the perpendicular bisector that appears in elementary education are reconceptualized when they are explored dynamically (see the task in the following section). Indeed, this concept becomes crucial for generating and exploring all conic sections studied at the high-school level (Moreno-Armella & Hegedus, 2009; Santos-Trigo & Ortega-Moreno, 2013). At this point, it is crucial that these reflections take place in a classroom with teachers. We cannot conceive of transforming education without the conscientious efforts of the teachers. This is strategic.

Let us begin now presenting some of the aforementioned vignettes and exemplars pertaining to ***mathematical ways of thinking*** for teachers.

A RECONCEPTUALIZATION OF THE PERPENDICULAR BISECTOR

Of course teachers, could describe this geometric object, but did they ***understand*** it? Had they used it in meaningful ways? These are questions we wanted to explore. For our work, we decided that ***understanding*** meant the moment when a cultural artifact (Trouche, 2004) (as the perpendicular bisector) became an instrument integrated with other cognitive instruments. Yet, this was rather restricted, so we searched for the moment the teachers in the classroom were *aware* of the instrument and they could use it to solve a task. This approach did cohere with the possibilities for active exploration within the digital environment. We offered the following task for exploration. Let us consider the segment AB and its perpendicular bisector (as shown in Figure 25.1).

At this point the idea was to use the perpendicular bisector as an *organizing principle* to explore conic sections. We suggested the construction of triangles with the third vertex C on the perpendicular bisector and then asked teachers to explore the locus of D (D is the intersection point of the perpendicular bisector of side BC and the bisector of angle CAB) as C travels

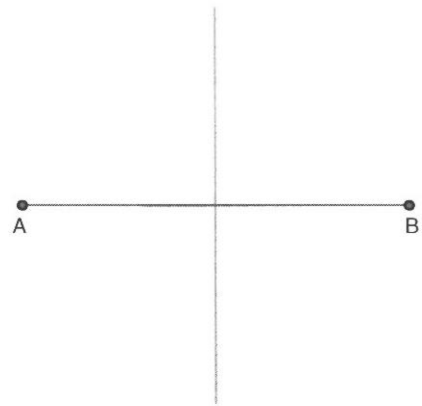


Figure 25.1 Perpendicular bisector.

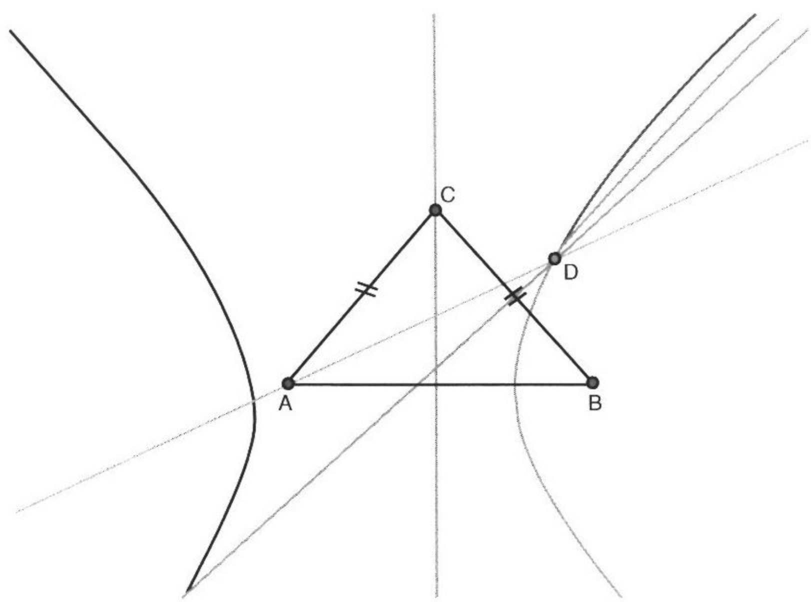


Figure 25.2 "Conic" section.

along the perpendicular bisector of AB (see Figure 25.2). Teachers got a locus that looked like a conic section. Now the problem began: Is it a conic section?

The figure does not do justice to what happened next. At that moment teachers were puzzled: The question was unexpected. They had worked with conic sections using the traditional analytic expressions. Now, where were the coordinate axes? After a while a teacher, Victor, realized they could try "to cover" the locus with a conic section passing through five points of the locus. The Geogebra dynamic system provides a command to draw a conic that passes through five points and teachers had used it extensively. Following this idea, teachers understood (after a mediating discussion) that the way to disprove the conjecture was ***legitimate***, as they had used something ***infrastructural***: the conic section passing through five points. They found the conic that disproved the result by dragging and rotating the vertices and the segments in the figure. This is crucial: they dragged, rotated the figure while ***preserving the underlying structure***.

We thought it was more productive to begin with a problem that would find a solution by means of a counterexample. The lesson learned was that dragging is a mediator for exploring.

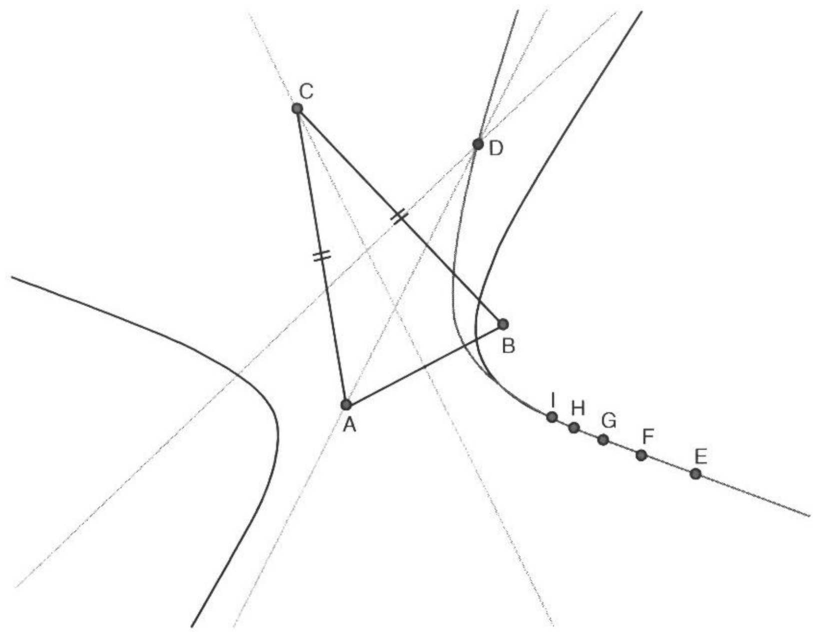


Figure 25.3 Another counterexample.

As a teacher, Laura, said, "If something is true you cannot *destroy it* by dragging it." The window to structure was opened. In the next session, Alvaro came with the construction shown in Figure 25.3.

His five points for the conic were *E, F, G, H*, and *I*. He began playing with the construction by moving Point I along the locus and observing the different conics.

The way to introduce the five-point construction as an infrastructural artifact in the digital medium was establishing the similarity with (i) two points determine one straight line; (iii) three points determine one circle. Then we discussed the fact that mathematics was embedded in the medium.

Discovering the Parabola

Teachers were very enthusiastic about the experience of solving a problem by means of a counterexample. The next time we decided to try a variant of the former exemplar. Instead of taking the bisector of angle *CAB*, we suggested that teachers work with the intersection point of the perpendicular bisector of *CB* and the perpendicular to line *b* at *C* as shown in Figure 25.4.

This time the locus is a parabola and the straight line b is the directrix; the focus is Point *B*. Teachers had already worked with the definition of the parabola as the locus of points equidistant from a line (the directrix) and a point (the focus). They identified *b* as the directrix and *B* as the focus—but what we did not expect was the fact that, to identify the locus as a parabola, they rotated the figure a right angle to the left, as shown in Figure 25.5.

They hid the segments and points that were not relevant for the original definition of the parabola as a locus. We were wondering why they had to rotate the graph as it was clear that the locus as shown in Figure 25.4 is a parabola. It was clear for us but *not* for them: The definition of parabola *always* comes with this graph (Figure 25.5), so an inertia is created due to the fact that the graphical representation of the conic reflect *our own body*, as when we draw on the slate. There is a sense of vertical and a sense of horizontal that are present when one tries to recognize a shape. It is clear that this event illustrates the *embodiment of knowledge* (Moreno-Armella & Hegedus, 2009).

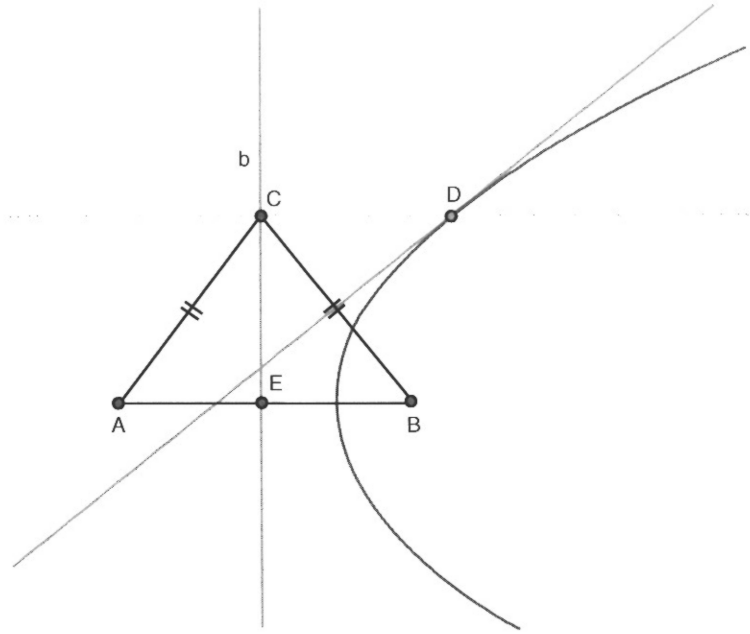


Figure 25.4 The locus of D as C travels on b.

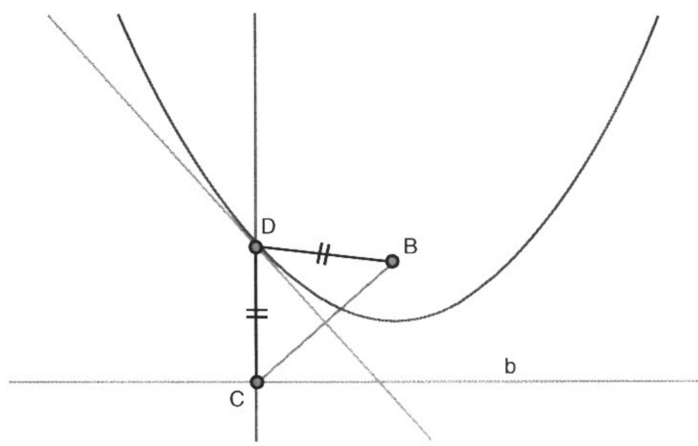


Figure 25.5 Identifying the locus.

Thus, simple mathematical entities such as triangles or circles can be represented digitally and become a platform or departure point to identify and explore more complex entities. This is the case with the following exemplar:

Looking for the Hidden Conics

Draw a circle with center C. On the circle we choose Point E and draw line CE. Then, we select Point F on line CE and draw the segment FG. We take the perpendicular bisector of FG. This perpendicular bisector intersects line CE at H (Figure 25.6).

We asked: What is the locus of Point H when Point E travels the circle? Figure 25.7 shows that the locus seems to be a hyperbola.

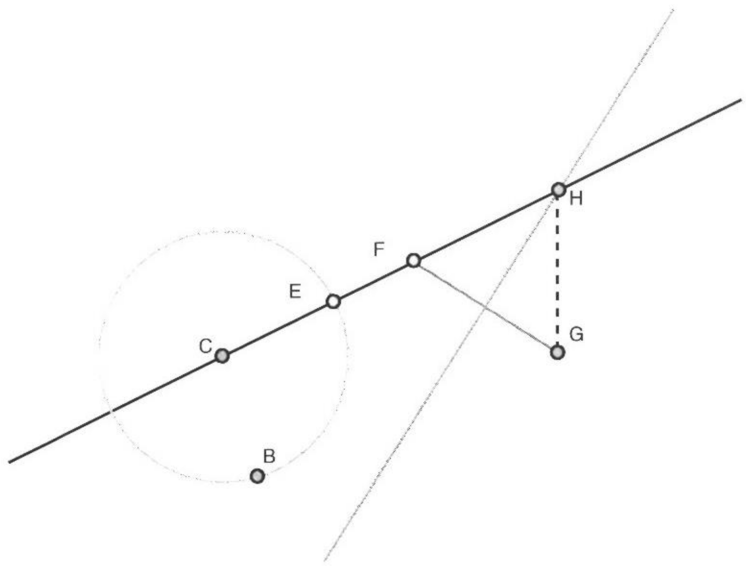


Figure 25.6 Dynamic triangle and perpendicular bisector.

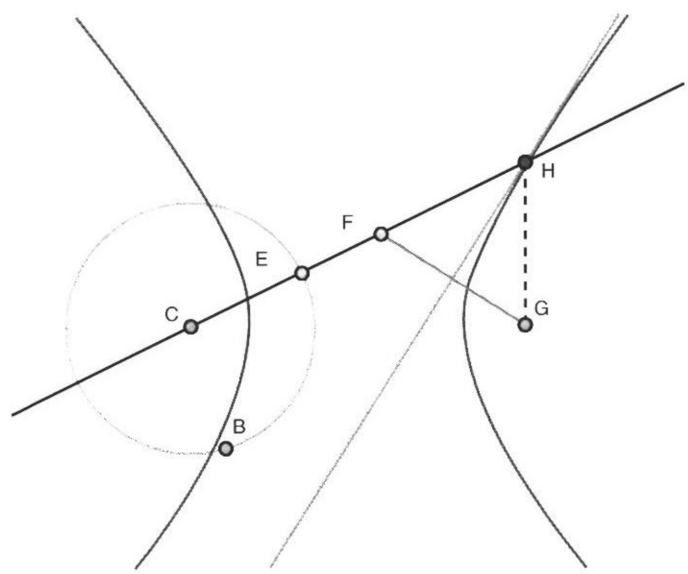


Figure 25.7 What is the locus of Point H when Point E travels along the circle?

Drawing the locus of H is an infrastructural affordance of the environment. In this context, it is a point-and-click action. Teachers were amazed with the environment's answer. Indeed, the locus *seemed* to be a hyperbola. But was it? At this time, the teachers were almost lost; they could not devise a plan of action. At our suggestion they measured the distances involved and found that the segment HF was congruent to segment HG. But—and this was Manuel's conclusion—that always happens because H is on the perpendicular bisector of FG! They drew the segment HG and things became clearer. It took another half an hour to write:

It is observed that (for every position of H):

$$d(C, H) - d(H, G) = d(C, F)$$

Because $d(H, F) = d(H, G)$. Consequently, the locus is a hyperbola.

We want to emphasize that the ability to drag points and observe the smooth morphing of the locus was instrumental for reaching the right conclusion. In this case the moving point was F. That made the teachers propose that C and G as the foci of the hyperbola. There is no doubt: ***Motion is worth a thousand pictures.***

It was visible that by moving Point G (this time F was fixed) different loci were obtained. We observed that they were moving G far from the circle, so we decided to ask: What happens if G gets closer to the circle?

We kept quiet for a while as they discovered that ***suddenly*** the hyperbola morphed into a figure that seemed to be an ellipse.

Teachers found astounding this sudden morphing into an "ellipse" when Point G got closer to the circle. They had ***proven*** after a while playing with the resources provided by the environment that $d(C, H) - d(H, G)$ was a constant equal to $d(C, F)$. Now, the morphed figure seemed to be an ellipse

It was not the difference but the sum: $d(C, H) + d(H, G) = d(C, E)$, a constant for every position of H on the perpendicular bisector of segment FG.

That made clear for them that the locus was, indeed, an ellipse.

Eventually, teachers came to perceive that the position of F on line CE has "the key" (their words) to decide if the conic was a hyperbola or an ellipse.

We thought, at that point, that it was time to simplify the construction by identifying Points E and F. Then we asked the teachers to figure out how to draw a tangent to the ellipse from any Point C inside this circle (see Figure 25.9).

We will omit this discussion, which completed a basic ***dynamic*** analytic geometry course, as we want to share a couple of Calculus examples that we discussed with another group. However, we consider important to offer some reflections based on the previous teaching-teachers experience before the Calculus exemplars.

A Brief Reflection

The Point F (see Figures 25.8 and 25.9) is a ***hot-point*** (Moreno-Armella & Hegedus, 2009) because if we keep fixed all other points, in these constructions, Point F controls the underlying structure of conics we can display. What is really central is that the environment provides these points in every construction. Emphasizing this idea made teachers aware that what we have on the screen are not simply dynamic drawings but geometric ***structures***. It was the movement that made the structure visible: the structure is hidden behind any particular rendition on the

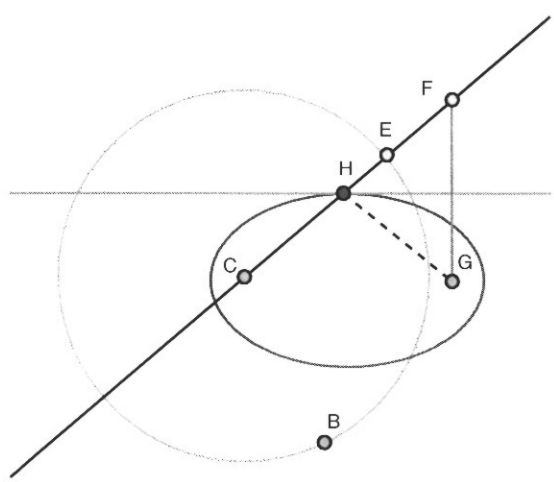


Figure 25.8 Ellipse with foci C, G.

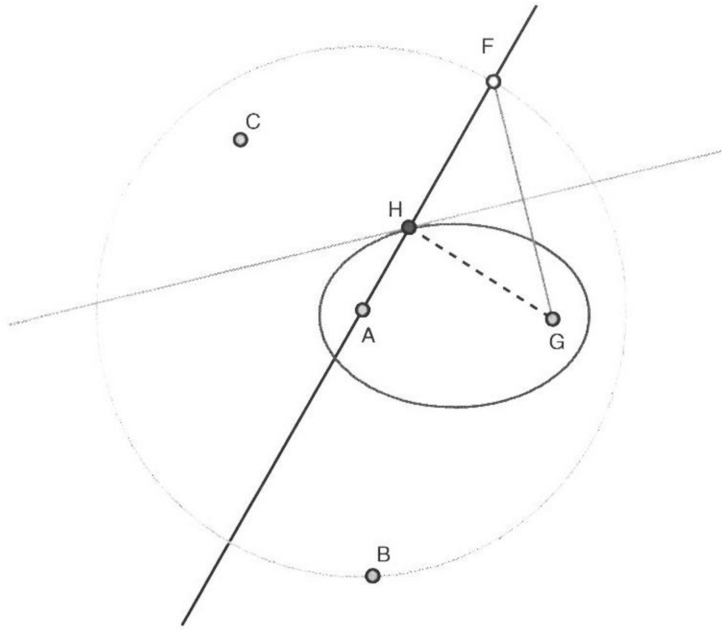


Figure 25.9 The new construction.

screen. One of the teachers, Laura, mentioned that the idea that the structure is visible through movement was similar to camouflaged objects: if a moth is standing still on a tree, then the bird (its predator) cannot see the moth, unless the moth moves. Of course, the similarity ends here as ***seeing the structure*** is a very complex cognitive process. What one sees, through the digital, executable representation is a conceptual entity (Moreno-Armella & Hegedus, 2009). At this point we pondered over the pertinence of going back to the discussion on the nature of mathematical entities. We found that working in a dynamic environment where learners could drag a figure, and modify the length of a segment, for instance, gave them an opportunity to explore the behavior of a family of objects within the same configuration.

In all the preceding examples, the basic geometric construction has been the perpendicular bisector. This construction is the ***organizing principle*** for exploring conic sections the way we chose to follow.

Action does not belong (exclusively) to the user and neither does it to the environment; both user and environment are actors and reactors. For instance, if we drag a triangle on the screen, it seems as if we are able to ***hold*** that figure with our hands and transform it. User and environment are, from the point of view of agency, *coextensive*. Thus, we speak of ***coaction*** between the user and the environment, not just between the user and the artifact. Coaction is the broader process within which an artifact is being internalized as a cognitive instrument. Yet, in the social space of the classroom there can be a collective actor. One participant can observe how another drives the technology at hand and then incorporate what she observed into her subsequent strategies. At the end participants can act and react to the environment in ways that are essentially different from their initial actions. We can learn ***from***, ***through***, and ***with*** the others. So the traditional triangle user-technology-task has to be enlarged: coaction becomes embedded in a social structure. Culture cannot be factorized from the technology appropriation processes, and technology cannot be factorized from culture.

TWO EXEMPLARS FROM CALCULUS

Digital media, with their *executable* representations (Moreno-Armella, Hegedus, & Kaput, 2008, p. 105), have transformed the traditional mathematics of change and accumulation.

There is a profound cognitive difference between applying a geometric transformation, ***on paper***, to rotate or dilate a triangle (where all the action takes place in human imagination) and applying that transformation through its executable version and perceiving it on the screen. Thus, variation, change, and accumulation are no longer restricted to the written version of Calculus. However, paper-and-pencil tradition cannot be ignored and left aside. We have to allow its representational redescription in terms of digital representations. In fact, there are many mathematical entities that can be redescribed, translated into digital environments and explored there. In our work with teachers we always intend to take advantage of digital representations for going deeper into the mathematics that we discuss and explore with them. In the next pages we will introduce two exemplars from elementary Calculus that have been the matter of intensive discussions in the classroom.

Area Approximation

Pierre de Fermat (1601–1665) solved, in a very original way, the problem of computing the area under a parabola: $y = x^p$, $p \neq 1$. Fermat began by subdividing the interval [0, 1] into an infinite sequence of subintervals with end-points of the form z^n, with $n = 1,2,3,\ldots$ and Z a fixed number $0 < z < 1$. That is, Fermat used an infinite subdivision of the interval by means of a geometric progression as suggested by Figure 25.10.

It takes some work to find a closed expression for the sum of the areas of the rectangles, but Fermat did it. Afterwards, Fermat's reasoning was to eliminate the error, that is, to fill the white triangles over the rectangles. At this point, we decided to stop the narrative (indeed, avoid the computations) and ask the teachers, taking into account their already gained experience with digital environments, how they could ***explain*** Fermat's result. They had some experience working with sliders; the answer came after a collective and very emotional discussion in the classroom (Vizgin, 2001). Let us see the next figures.

Figure 25.11 on the left shows a fixed value of $Z < 1$, and six rectangles. The slider is used to control the number of rectangles. Figure 25.12 on the right shows six rectangles, but now the value of Z is closer to 1. One can observe that the process of approximation depends not

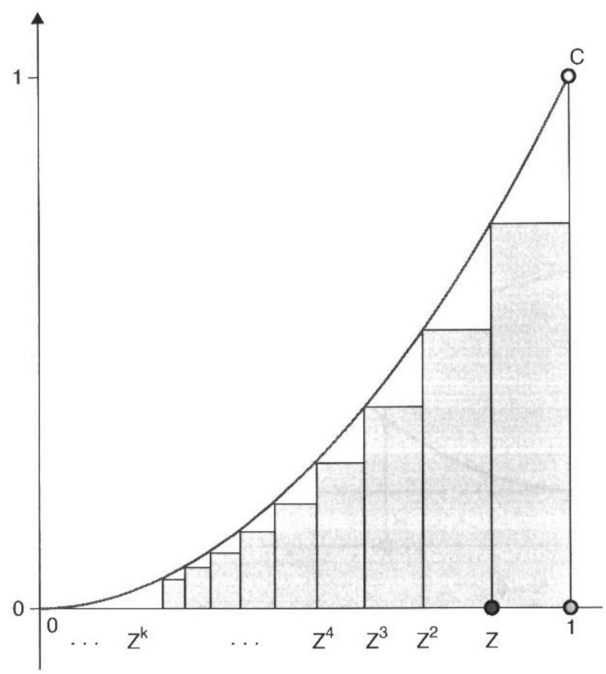


Figure 25.10 Infinite subdivision.

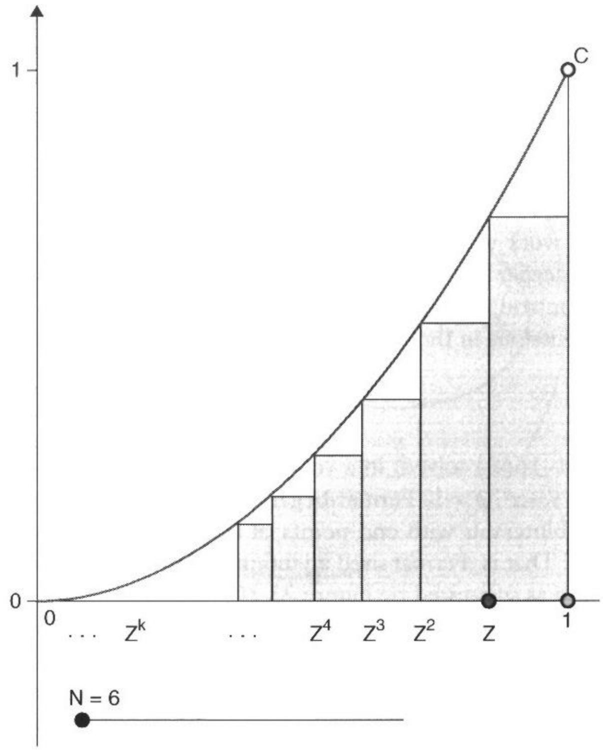


Figure 25.11 Increasing the number of rectangles.

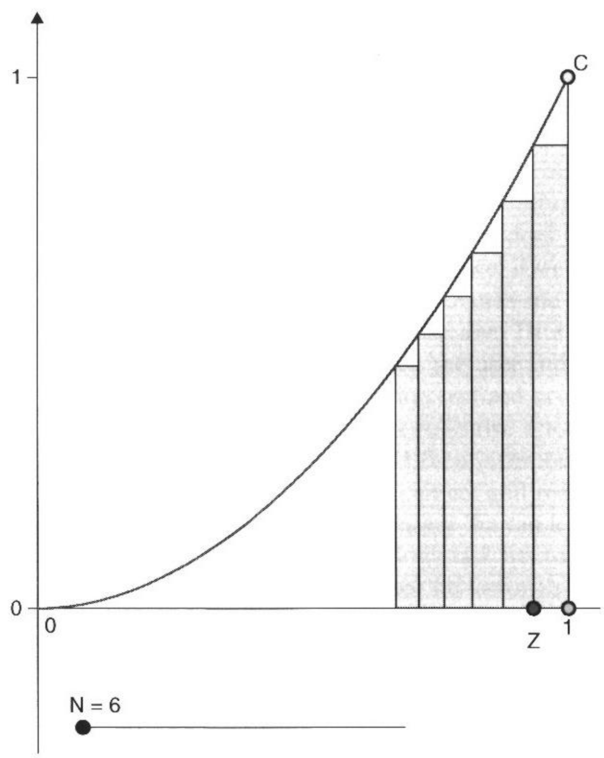


Figure 25.12 Obtaining a better approximation.

only on the number of rectangles (***in principle***, there is an infinite number of rectangles) but on how close to 1 is Z. If Z is closer to 1, then we need more rectangles.

We certainly believe that the teachers were able to understand the basic idea behind an approximation process, notwithstanding the contextual constraints. We have called these kinds of particular contexts ***domains of abstraction***: There is something general "hidden" below the context in question (Moreno-Armella & Sriraman, 2010, pp. 224–225).

It became tangible for the teachers that the environment is full of "treasures" (they used this word), affordances, let us say, that enable the user to express her/his mathematical ideas. There is no neutral artifact, no neutral environment. Each artifact ***drives*** the actions of the user (individual or collective) and is ***driven*** by the user in a coextensive process that leaves no one unchanged. As a cognitive agent, the user eventually incorporates the artifact to his/her cognitive resources. That is what we do when add two numbers: we do not *see*, anymore, the decimal notation system that after years of schooling has become incorporated as a cognitive instrument.

Among the treasures the teachers became aware of, sliders and dragging were instrumental for their mathematical thinking: These became instruments to deal with and control continuous and discrete variation.

One of our goals was to help the teachers to develop conceptual and computational fluencies. That is, the tool's affordances are vehicles to represent and explore concept meaning and its uses in problem-solving activities. We believe this is possible if teachers have at their hands the mediation of dynamic, digital environments.

It is important to realize that blending mathematical ideas originally developed in static media with their digital redescriptions has the potential to open windows into a new mathematical culture in the classroom.

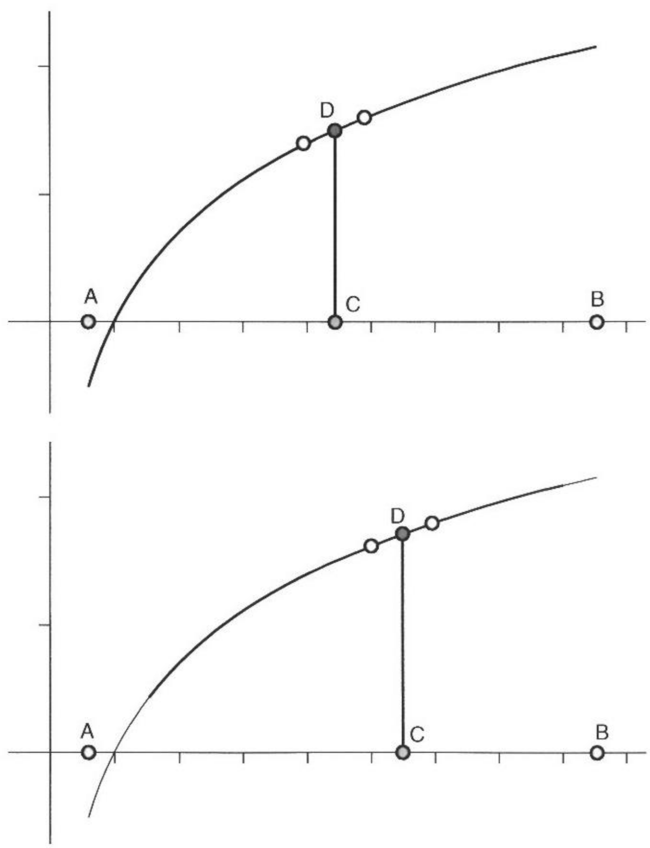


Figure 25.13 A tangent segment.

The Visual Derivative

One of the main obstacles to developing a sound vision of the derivative is *not* being able to conceive of it as a number. Once the derivative has been introduced as the slope of the *tangent line to the graph*, at one point, in most cases this tangent is used to locate maxima or minima of the function. This is done by following a mechanical procedure, almost a mantra: find the derivative; make it equal to zero . . . and so on. But the important step of finding the derivative *and trying to understand what it says about the function globally* is almost never taken. This is the inertial effect we talked about previously. It is part of school culture.

We decided to discuss this issue, taking as a starting point the graph of a function and the tangent line at a point of the graph. Then we took a small segment of the tangent line around the point of tangency as shown in Figure 25.13. The idea we wanted to introduce was that a short—a very short indeed—segment of the tangent line around a point of tangency could generate the graph of the function. Next we dragged the segment (activating the trace for the segment) and we produced the figure on the right in Figure 25.13.

Then a discussion began about the meaning of *close* when we say that the tangent line is the best approximation to the function around a point. After a while we proposed that the teachers discuss the following situation in which we hid the graph of the function but kept visible the tangent segment as in Figure 25.14.

Then, we dragged the segment (with the trace active) and showed that we could recover the curve, the whole curve. Some teachers were amazed, and then one of them essentially asked: "What does the tangent know?" Another replied: "It does not have to know anything, because

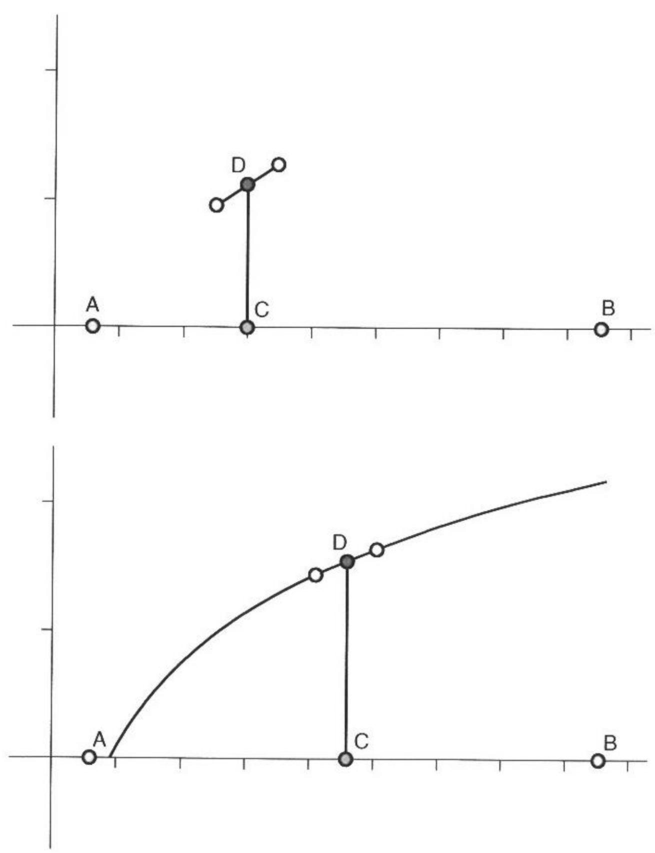


Figure 25.14 Recovering the graph.

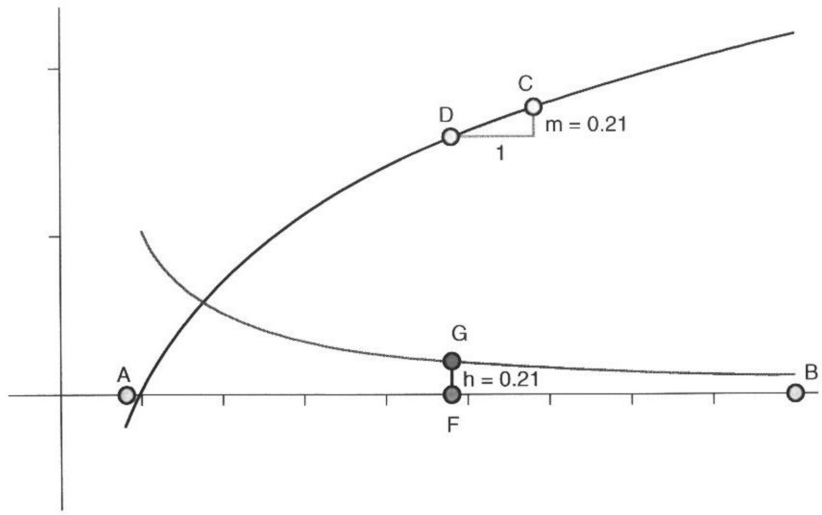


Figure 25.15 Graphing the derivative.

the curve is hidden but not erased." So, in a sense dragging the segment was a way to ***uncover*** the hidden curve. We thought this discussion was very valuable indeed; something deep was floating in the atmosphere of the classroom. We thought the time was right for making explicit a seed about the fundamental theorem of Calculus. We have simply to extend some ideas that were already under discussion with the teachers. If you have the function then you have the tangent lines—and reciprocally, if you have the tangent lines you can recover the function.

Figure 25.15 illustrates how we tried to animate this discussion.

This time, our emphasis was on establishing that each time you have the function, in fact you have two functions: the one you already have, and the derivative function that maps the behavior of the original function. In these tasks, we were trying to emphasize the conceptual fluency beyond the operational fluency that the teachers were more familiar with.

It has become clear from the vignettes and examples we have previously outlined, that our math intuitions rely in very specific ways on action and motion, and that the digital environment has provided our group of teachers a great service: it has helped them to transform metaphorical thinking on motion and action into sound cognitive instruments.

SIMULATION AND MODELING: ANOTHER EXEMPLAR

The use of digital technology also plays an important role in constructing dynamic models of tasks or situations that involve realistic contexts. For example, Figure 25.16 shows a truck that is approaching a certain underpass where there is a sign indicating the maximum clearance (the height of the bridge). The bridge is located just at the base of a descending roadway (Figure 25.16). What data and conditions do we need to know in order to figure it out whether the truck could clear the bridge? What is the effect of the inclination of the roadway on the height of the truck when passing under the bridge? (A similar task appears in NCTM, 2009 and Santos-Trigo and Barrera-Mora, 2011).

High-school teachers worked on this task. Initially they spent significant time making sense of the task statement and discussing questions regarding dimensions of the truck, wheel positions, height of the bridge, etc. At this stage the goal was to think of a two-dimensional representation of the problem in terms of mathematical objects (lines, rays, angles, rectangles, circles, etc.). Figure 25.17 is a simplified representation of the roadway, the bridge, and the inclination angle of the roadway.

Figure 25.16 The truck entering into an underpass.

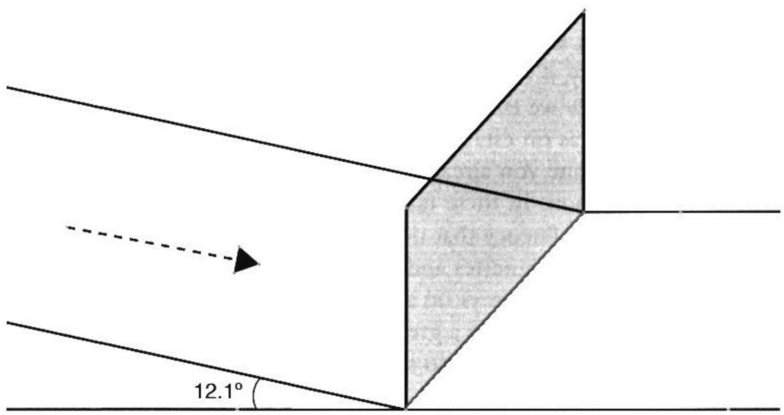


Figure 25.17 The roadway, the bridge, and the inclination angle of the roadway.

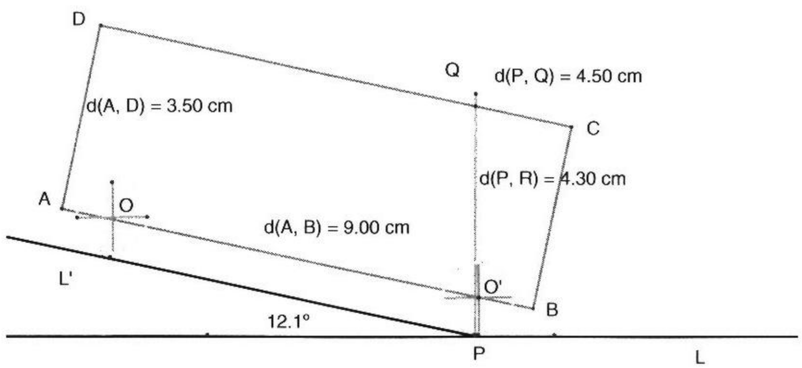


Figure 25.18 A task model constructed through the software.

During the process of constructing a dynamic model of the situation, teachers relied on a set of considerations and assumptions to represent information embedded in the task in terms of mathematical objects. For example, the roadway was represented through two intersecting lines, L and L', the wheels were represented with circles, the truck box with a rectangle, and the height of the bridge with a segment. In addition, when the positions of the wheels were all on the tilted position or on the horizontal position (after the back wheels have crossed the bridge), the sides AB and DC were parallel to line L' and, after the back wheels crossed the bridge, to line L (the roadway; see Figure 25.19). Likewise, it was assumed that the truck's tilting effect, which might produce a shifting load on the truck's wheels, does not distort the height of the truck.

In the model (Figure 25.19), Point M was a mobile or pivot point, Point Q was chosen at a fixed distance equal to 4.5 cm from P (height of the bridge). It is observed that by moving point M along line L, the height of the truck, measured as the distance from Point P (bridge initial point) to Point R, the intersection point of the perpendicular to line L through Point P and the upper side of the rectangle (segment DC) (Figure 25.19) changes depending on the position of Point M. The trace left by Point R when Point M is moved along line L represents the graph of the trunk height variation as a function of the position of Point M. Under these

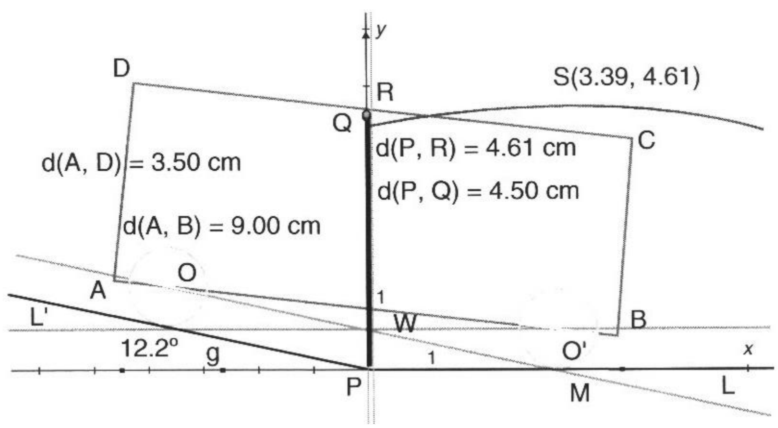
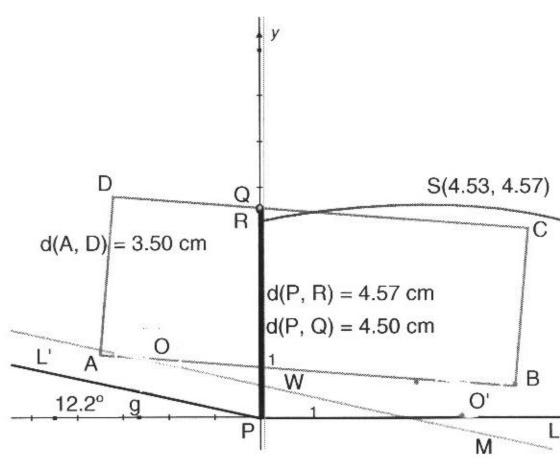


Figure 25.19 Graphic representation of the variation of segment PR as Point M is moved along line L.



1	0.24	4.33
2	0.34	4.35
3	0.45	4.36
4	0.47	4.37
5	0.71	4.41
6	0.97	4.44
7	1.13	4.47
8	1.63	4.52
9	1.95	4.55
10	2.21	4.57
11	2.47	4.58
12	2.68	4.59
13	2.92	4.60
14	3.24	4.61
15	3.68	4.60
16	3.74	4.60
17	4.03	4.59
18	4.53	4.57

Figure 25.20 Three representations of the problem: the position of the truck, a table showing the variation of the height, and its graphic representation.

conditions, the teachers visualized that there was a position for Point M where the height of the truck reaches a maximum value (Figure 25.19). Thus, a truck with a height of 4.5 meters could not clear the bridge, since at one position of M its height is larger than the clearance under the bridge. In other words, the height of the bridge must be larger than 4.61 meters in order for the truck to clear.

Teachers discussed the fact that the use of the tool does not require expressing symbolically the relation between the position of the truck and its height. It was sufficient to relate the position of the truck's front (Point M) with the corresponding height (Point S) and later the locus of S when Point M is moved along line L (the roadway) generated the graph of the variation of the height. By moving the Point M along the line L, a table including values for different positions of the point and the associated height of the truck can be produced (Figure 25.20).

FINAL THOUGHTS

In 1996, world chess champion Garry Kasparov played a match against Deep Blue, an IBM supercomputer. Kasparov wrote in *TIME* magazine that he could feel, even *smell* a new kind of intelligence across the table.

After almost 17 years, Kasparov's story seems up-to-date; this new intelligence shapes our actions and behaviors. The zeal that let our community to face the challenge of thinking about thinking has been moving indeed. Today, this issue has come to the fore because the presence of digital technologies has made clear that we cannot restrict intelligence to "its confining biological membrane" (Donald, 1991, p. 359).

The *externalization of memory* that inaugurated this momentous stage developed into symbolic technologies. But even if one could feel an intelligence sitting on the page of a novel, that was human intelligence indeed.

Kasparov's feelings had a different source. Societies are already (or will be sooner than later), saturated with the presence of visible and invisible computers. Not all of them play chess but some are able to control the flight of a huge airplane across the Pacific. Others can give us the location of the restaurant we are looking for, or compute a complex mathematical model that we human beings cannot compute with static symbolic technology alone.

If the power of digital technologies is broadly tangible, there is no reason to expect they will not have as well a profound impact at the level of formal education. Educators have to cross that Rubicon and understand that the executable, symbolic, representations are key to making even more tangible the zone of potential development of social, distributed intelligence. Indeed, intelligence is a network phenomenon and we have to conceive of it globally, seamlessly, in a move that includes all kinds of intelligences, such as that Kasparov caught a glimpse of across the table.

What about schools? We might say that old habits die hard—but it is not just a matter of habits, it is more a matter of transformation of cultures. The new classroom with the possibility of sharing an expressive medium, like the digital environment, can help us organize open mathematical discussions and foster a continuous reflection within a social space in permanent evolution. In this space, the meaning of mathematical entities evolves with the opportunities to directly manipulate them.

Mathematical entities, as explained previously, are only indirectly accessible through semiotic representations (Duval, 2006). Consequently, the only way of gaining access to them is using, for instance, words, symbols, expressions, or drawings. But no representation exhausts the represented entity. Nevertheless, any mathematical representation has such a crystallizing impact on how mathematical entities are experienced that when we work with it, we have the feeling of working inside a Platonic mathematical reality. But this is only an illusion that lurks beneath the surface. Mathematical reality is a human reality even if it is a virtual one: one cannot forget that humans have the power to extend their world of experience symbolically.

Closer to our professional interests is the ***mode of existence*** that teachers have experienced whilst working with the dynamic geometry environment and when they analyzed and discussed a design activity whose goal was to construct a dynamic model involving a truck approaching an underpass. In all these activities, it is made tangible that we can explore and experiment on dynamic representations of mathematical entities as if they were material objects (Santos-Trigo & Reyes-Rodríguez, 2011). In fact, the executable nature of dynamic representations enables the learner to continuously modify those representations while preserving their structural features. This reflects a profound difference from the static representations of traditional mathematics at school. The kind of intelligence living in the executable representations extends human action with digital artifacts into the social space of the classroom. The end result of this process is an instrument loaded with the intelligence shared in the classroom. In practice, it took long weeks for the teachers to master and embody new ways of interacting with the virtual reality of digital entities. No artifact is epistemologically neutral; consequently, there is a disruption in the taken-for-granted aspects of what it means to think mathematically in digital contexts. In this view, an instrument—that is, the internalized artifact—is a template for action. It is relevant here to mention that, with the instrument, the learner can explore new landscapes of mathematical validation. In fact, the notion of ***theorem in motion*** embodied in the dynamic digital environment comes to the fore; this is how we conceive of it. Then, reconsidering the transformation of static entities through executable representations, we are opening a window to new mathematical entities whose proper ecology is the digital. But it is not the search of the object per se what moves us as researchers, but the search for new ways of thinking.

We expect that the mathematics of change and variations, through their digital embodiments, will contribute to a substantial gain in students' development of conceptual understanding and computational fluency.

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