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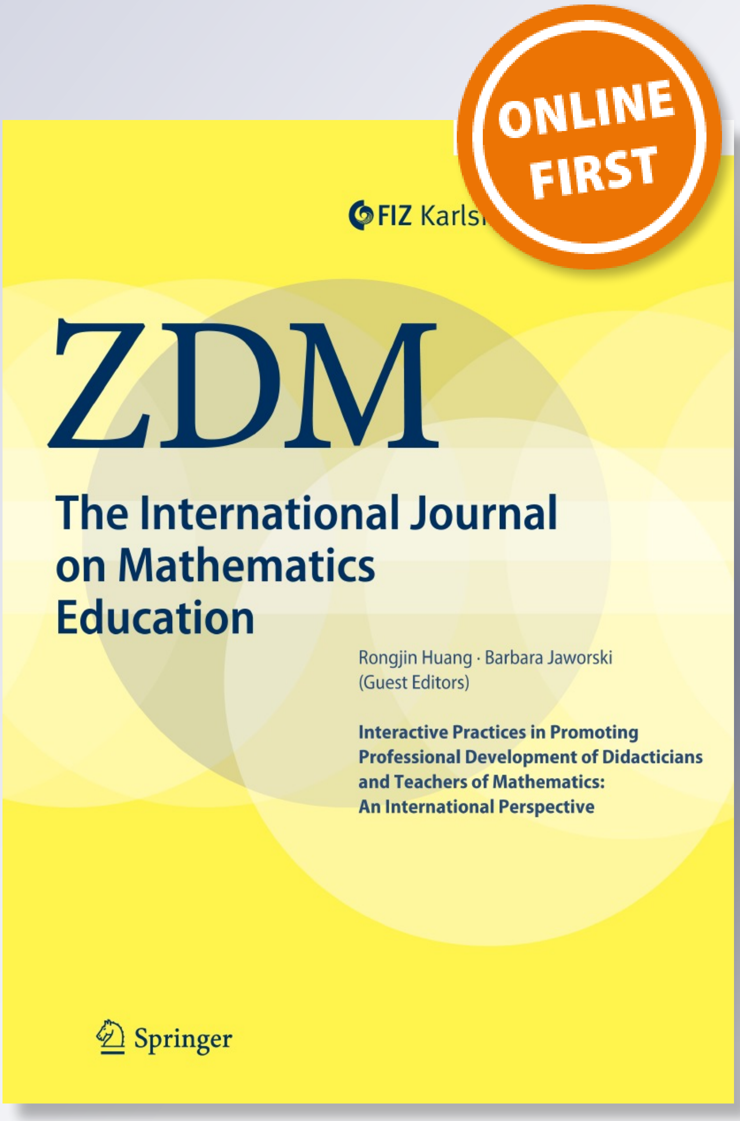
ZDM
The International Journal on
Mathematics Education

ISSN 1863-9690

ZDM Mathematics Education
DOI 10.1007/s11858-014-0580-4



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ORIGINAL ARTICLE

An essential tension in mathematics education

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Accepted: 10 April 2014
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Abstract There is a problem that goes through the history of calculus: the tension between the intuitive and the formal. Calculus continues to be taught as if it were natural to introduce the study of change and accumulation by means of the formalized ideas and concepts known as the mathematics of ε and δ. It is frequently considered as a failure that "students still seem to conceptualize limits via the imagination of motion." These kinds of assertions show the tension, the rift created by traditional education between students' intuitions and a misdirected formalization. In fact, I believe that the internal connections of the intuition of change and accumulation are not correctly translated into that arithmetical approach of ε and δ. There are other routes to formalization which cohere with these intuitions, and those are the ones discussed in this paper. My departing point is epistemic and once this discussion is put forward, I produce a narrative of classroom work, giving a special place to *local* conceptual organizations.

Keywords Variation · Infinitesimal · Infinite · Archimedean principle · Symbol · Intuition · Formalization · Continuum · Limit · Derivative

1 Introduction

Change is permanent and omnipresent. Human efforts to understand a world of variation and accumulation go back well into the past of our species. The motion of the planets, the evolving weather, the growth of a stream are just a few examples of early human experiences that have contributed to the awareness of change. Symbols have paved the way to translate human experience into symbolic models. To deal with quantity, shape, and change (at the social and personal levels), we needed symbols, and this is how mathematics was born. Of course, I am not going to tell the full story now, even though some pieces of it will be necessary for further reflection.

Calculus is the mathematics of variation and accumulation. It constitutes a substantial chapter, perhaps the most substantial, of the mathematics developed during the 17th and 18th centuries. The study of speed and distance traveled by a moving object led to the classical models of the tangent to a curve and the area under a curve. The power provided by the methods of analytic geometry was instrumental to configure this setting and to provide an initial context for these two problems—the tangent and the area. Problem after problem, example after example, Fermat, Cavalieri, Wallis, and many others devised ad hoc strategies to solve them and, at the same time, caught a glimpse of the generality behind the particular problems they could solve. It was necessary to wait for Newton (1642–1727) and Leibniz (1646–1716) to crystallize what was in the making: the *fundamental theorem* establishing the deep link between the tangent and the area under any given curve.

At every stage of the history of ideas, the accumulated work of a generation opens new vistas to their followers. It is as if what one generation conquers, casts a pale light on the road that their followers transform into solid ground. Perhaps this is what Newton meant when he said that if he had been able to see further, it was by standing on the shoulders of giants. Newton established a version of the fundamental theorem, but he did much more indeed. He was the man who *by strength of mind almost divine, explored the course of the planets, the path of comets, and*

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Published online: 27 April 2014

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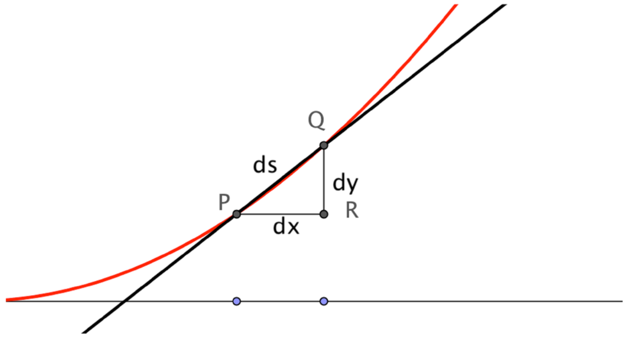

Fig. 1 Leibniz triangle

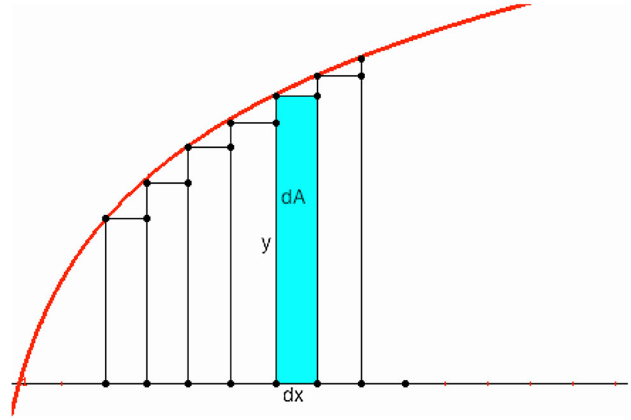

Fig. 2 Infinitesimal area

the tides of the seas. These words can be read in front of his monument in Westminster Abbey. They reflect a conception of mathematics that was not Newton's exclusively: mathematics as a mirror of nature. Euclid's geometry was organized around axioms that translated basic experiences from physical space. And Galileo in his work *The Assayer* famously wrote that the great book of nature had been written with mathematical characters (Forinash et al. 2000).

Mathematics is a symbolic version of nature built on basic intuitions, observations about the book of nature. Newton wrote his *Principia* in the language of Euclidean geometry and translated his physical intuitions into this language.

Calculus does not have one father, it has two. Leibniz was a qualitatively different thinker from Newton. Leibniz was most interested in a universal characteristic, a kind of language that could help to resolve any dispute if accepting this invitation: *let us calculate* would be the magical words. His love for language was, as well, instrumental in his version of calculus. He conceived of this characteristic as a "sensible medium that would guide the mind as do the lines drawn in geometry" (Edwards 1979, p. 232). Leibniz's was a calculus where the notion of infinitesimal played a key role. A curve, for instance, was considered as a polygon whose sides are infinitesimal segments of straight lines.

To explain how to find the tangent line, Leibniz, in his first publication of calculus, wrote (as cited in Kleiner 2001, p. 146; Struik 1969, p. 274):

> We have only to keep in mind that to find a *tangent* means to draw a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the place of the *curve*. This infinitely small distance can always be expressed by a known differential like ds.

The Leibniz triangle (Fig. 1) is the heart of infinitesimal calculus. Following this train of ideas, the area under a curve can be thought as the infinite sum of rectangles with infinitesimal base, as suggested by Fig. 2.

The infinitesimal area dA of any rectangle with base dx is ydx. Consequently,

$$\frac{dA}{dx} = y$$

This is essentially the fundamental theorem of calculus. In a very intuitive way, variation and accumulation appear deeply intertwined. The geometrical context provides the meaning for this profound fact. In addition, this example illustrates Leibniz's excellent notation. This notation is like a mirror that reflects faithfully the ideas and, from that instant on, it will not be possible to distinguish between the symbols and the ideas embodied in those symbols. Having Leibniz in mind, Jacques Hadamard wrote: "The creation of a word or a notation for a class of ideas may be, and often is, a scientific fact of very great importance, because it means connecting these ideas together in our subsequent thought" (Edwards 1979, p. 89).

Leibniz could not ignore the nature of these infinitesimals: so similar, so close, but different from zero. In a product such as xdy a finite quantity x is multiplied by an infinitesimal quantity dy, which means that the rules of arithmetic have been extended to an *enlarged* numerical field. Leibniz and his followers based the development of calculus on these extended numerical rules and, on several occasions, Leibniz tried to explain the gist of the matter. In a letter to Bernard Nieuwentijdt in 1694 referring to infinitesimals, he wrote: "It will be sufficient simply to make use of them as a tool that has advantages for the purpose of calculation, just as the algebraists retain imaginary roots with great profit" (as cited in Edwards 1979, p. 265).

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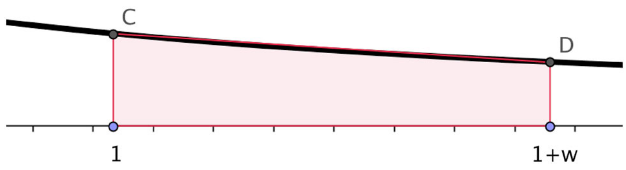

Fig. 3 Logarithm as area

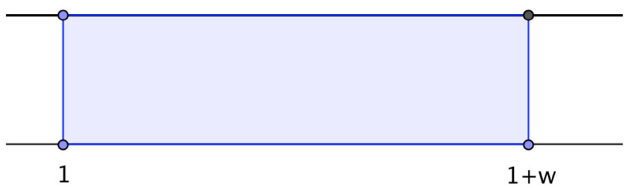

Fig. 4 Rectangular area

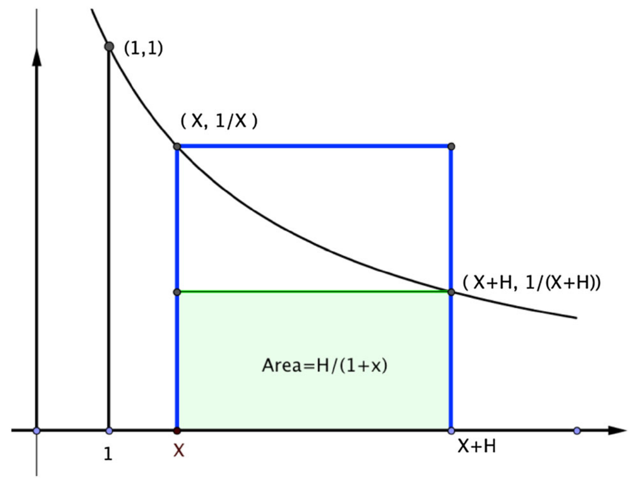

Fig. 5 Area inequalities

It seems that Euler (1707–1783) was able to penetrate Leibniz's intellect. Browsing through his book *Introduction to Analysis of the Infinite* (1748/1988) one gains the conviction that Euler profoundly understood the secrets of the infinite. In this book, Euler studied the logarithmic function. The area under the hyperbola $y = 1/x$ between 1 and any number $x > 0$ is defined as the natural logarithm of x. Now, to calculate the area under the hyperbola between 1 and $1 + w$, where w is an infinitesimal, one has to take into account that the function in that interval will vary infinitesimally (Fig. 3).

If w is an infinitesimal, then what you see appears through the *microscope for infinitesimals* as in Fig. 4.

The area between 1 and $1 + w$ is, on one hand, $\log(1 + w)$ and on the other, w. Consequently,

$$w = \log(1 + w)$$

Or, equivalently,

$$e^w = 1 + w$$

If x is a *finite* number and N an *infinite* (natural) number, then x/N is an infinitesimal. Let $w = x/N$. We can write:

$$e^x = (1 + \frac{x}{N})^N$$

With a bit more of work, expanding this binomial gives

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

We are extending the domain of numbers by including infinitesimals and infinite numbers. However, looking at the last expression one observes that infinitesimals and infinite numbers have disappeared. Infinitesimals and infinite numbers are thus artifacts whose purpose is calculation—just as the algebraists of the fifteenth century "retained imaginary roots with great profit."

To face the problem of teaching, a modest dose of systematization is included. Refining the intuitive understanding of calculus is possible with the help of *local organizations*. I define local organizations as environments where a general idea is embodied in a particular context, more familiar to the students/teachers, enabling them to enhance their understanding of basic ideas. Let us discuss an example in a visual context: how to obtain the derivative of the logarithmic function.

The difference $\log(x + H) - \log(x)$ can be represented as the area between x and $x + H$ under the graph of the function $y = 1/x$ (Fig. 5).

This figure makes it easy to understand the inequalities among the three indicated areas and this leads to:

$$\frac{1}{x + H} < \frac{\log(x + H) - \log(x)}{H} < \frac{1}{x}$$

As H becomes increasingly small, one can understand the soundness of the conclusion:

$$\frac{d}{dx}(\log(x)) = \frac{1}{x}$$

(We omit here the discussion of the case $0 < x < 1$.) Evidence is not necessarily formal proof, as this example illustrates. Understanding, rather, comes from the familiar geometric context provided by the figure.

We can proceed directly using Euler's idea that the segment of the curve between x and $x + H$ can be taken to be horizontal if H is an infinitesimal increment. Using the local flatness of the graph, we have Fig. 6.

That is, the area of the rectangle represents the difference:

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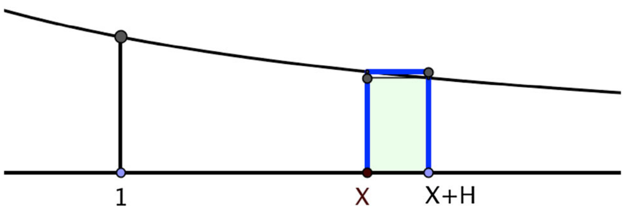

Fig. 6 Infinitesimal areas

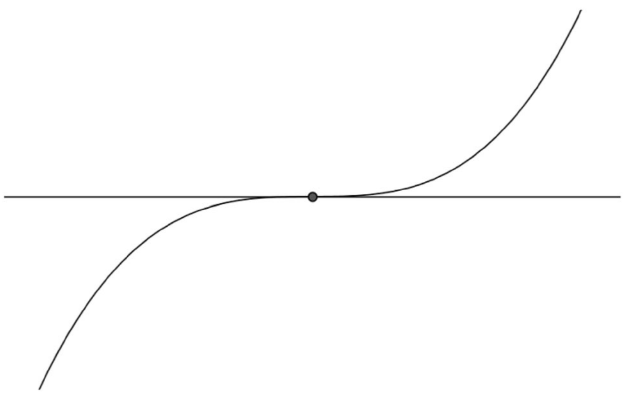

Fig. 7 Inflection point

$$\log(x + H) - \log(x)$$

Consequently, as the height of the rectangle is $1/x$, then

$$\log(x + H) - \log(x) = \frac{H}{x}$$

And as H is an infinitesimal,

$$\frac{d}{dx}(\log(x)) = \frac{1}{x}$$

One of the goals has been to identify a familiar idea or an intuition around which mathematical thinking can be activated. The problem of drawing a tangent to a curve represents one of the important conceptual models of calculus. To explore this problem one can take profit from the analytic representation with a procedure that embodies the action: *take the difference quotient*. The outcome is the "tangent object" but now, according to the analytic representation, the tangent object can "discover" the inflection points (Fig. 7), something that goes beyond the intuitive notion of tangent.

This problem has been studied with considerable depth. For instance, Biza (2011) has studied the cognitive conflicts that students undergo when they confront their global image of tangent with the local tangent at an inflection point. Her analytic approach was complemented with the digital exploration mediated by the zooming facility of the software employed, EucliDraw. There are other similar studies, for instance Maschietto (2008), who uses the term *micro-straightness* in the context of the graphing calculator. These studies, among others, have proved that magnifying the graph of a function, locally, is a powerful visual approach to study the notion of tangency.

The analytic re-description of the tangent line transforms the geometric, intuitive, image of the tangent line. Indeed, this analytic version does not prevent the tangent line from cutting the curve at the inflection point. How can the tangent line go through the curve? This possibility is not present in the original conception of the tangent. Now, this conception undergoes drastic modifications due to the exploration of such a situation by means of the analytic representation that forces us to accept the more general version of the tangent (Moreno-Armella 1996, p. 635).

In October 1979, Professor André Weil spoke at the University of Rochester on the life and work of Leonhard Euler. As Euler's translator John Blanton explained, Professor Weil was trying to convince the mathematical community that students would profit much more from Euler's book than from the modern texts of calculus (Euler 1748/1988, p. xii). The advice given by Weil was taken as one of the points of departure for my study.

Indeed, the ideas and mathematical formulas introduced so far in a very condensed way have made it feasible to organize a set of introductory lectures and discussions in the classroom with our students/teachers (in-service teachers from secondary level at graduate school) around some basic ideas of calculus. I have dealt with these ideas in an informal, intuitive setting. In the following two sections I present some critical reflections that will offer a more substantial perspective of the emerging tensions between the intuitive and the formal approaches in the teaching of calculus. Then, in the last section, I suggest how to enhance the ideas discussed for teaching so far, with the use of digital technology mediated by GeoGebra in this case. I hasten to say that neither approach is conceived as ancillary to the other. Leibniz's notation using infinitesimals is an action notation system and GeoGebra provides an executable notation system bringing motion to the screen.

2 The persistence of intuition

There is a principle of continuity implicit in the ideas offered in the previous section: everything is infinitesimally different from what it was an instant before. That is how one is led to conceive of a number $A + b$, with b being an infinitesimal, as equivalent to A itself in a computation; or of a smooth curve as a polygon whose sides have infinitesimal length. These are valuable ideas for teaching because they have a clear intuitive meaning. They

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offer to the student a new route to results that either had remained ill-understood or were disregarded. Through their practice, students will gradually become familiar with an increasingly refined version of the mathematical ideas pertaining to calculus. Working at the informal, intuitive, level entails working closely to the students' world of experiences. Eventually, a window will be opened offering the opportunity to reflect and discuss, in the social space of the classroom, the nature of mathematical entities.

This approach resonates with the words of Klein when, in 1896, he wrote:

> I have to point out most emphatically… that it is not possible to treat mathematics exhaustively by the method of the logical deduction alone, but that, even at the present time, intuition has its special province…. Logical investigation is not in place until intuition has completed the task of idealization. (p. 242)

Of course, Klein was not rejecting the need for formalization. He was, rather, making a pedagogical statement of value. He continued defending this position and, years later, in his celebrated book *Elementary Mathematics from an Advanced Standpoint* he discussed, anew, the need to balance the role of deduction and the role of induction in infinitesimal calculus and emphasized that sense perception was a great heuristic aid (Klein 2004, p. 208). From today's viewpoint, one would say that Klein was coming close to an embodied theory of cognition. As the psychologist Merlin Donald (2001) has written:

> The conscious mind may have reinvented itself and greatly extended its reach in language, but it has never lost its vestigial roots in embodiment. On the contrary, although human consciousness may have had to accommodate itself to the emergent symbolic structures of complex culture, refining, nurturing, and reflecting upon these structures as it expanded its own powers, it has always referred back to its roots in the physical self. (p. 137)

In his book, Klein (2004, p. 214) explicitly discusses the tension between the intuitive and the formal approaches when he states that although scientific mathematics is deductive, an essentially different conception of calculus, based on infinitesimals, has been running parallel with the former through the centuries. However, the loss of credence of infinitesimal calculus did not result from detected internal contradictions of a numerical system that included infinitesimals; rather, it was caused by the impossibility to legitimate infinitesimals as real mathematical objects according to new standards of rigor. Similar stories can be told of the acceptance of imaginary numbers, negative numbers, or irrationals.

Abraham Robinson (1966) finally overcame the crisis of the infinitesimals—but from an extremely formal approach that I do not follow in my teaching. From then onward, the classical results of infinitesimal calculus explained in the previous section could be proved formally and not simply intuitively explained. This is the case for instance with the theorems of Euler. Now they have been proved "correctly," but within the context of a different mathematical culture. We discuss in the classroom Euler's approach to this result:

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots$$

Was it true when Euler obtained it? What does the new (formal, rigorous) proof add to the result? These are delicate questions, but knowing something about the evolving life of a theorem is a piece of knowledge a teacher must possess. Success in mathematics does not imply success in teaching mathematics. This has been discussed many times but, apparently, it is a misunderstood assertion. When the school does not hear the voice of the students, frequently uttered too low, the result is to expel students from the Eulerian paradise of calculus—to paraphrase Hilbert. In response to this, Nikolai Luzin writes:

> What Weierstrass, Cantor, and Dedekind did was very good. That is the way it had to be done. But whether this corresponds to what is in the depths of our consciousness is a very different question. I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of the "justifications" and their "proofs." (As cited in Demidov and Shenitzer 2000, p. 80)

The problem here is that the formalization of the ideas of the mathematics of change and variation does not correspond to the so-called arithmetization of calculus. This results in a kind of cognitive impasse for mathematics education. In their learning process, primary experiences from variation and accumulation are instrumental for the students. This kind of experiences provides the root of meanings.

Calculus textbooks should provide students with the conceptual tools to study the mathematics of change but instead they offer a formal narrative whose tacit structure banishes change. This creates a rift between the intuitive, embodied ideas of variation and accumulation (their symbolic notation is controlled by basic motion metaphors, for instance *approaches*, *oscillates*, *tends, infinitely close to*) and the formal structure whose telos is quite different: to create a justification structure based on the static, rather frozen, approach corresponding to the program already advocated by Bolzano (1781–1848) who in 1817,

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following his theory of science, wrote that the concept of time and motion were as foreign to general mathematics as that of space (as cited in Bottazzini 1986, p. 98). There are historical reasons for the persistence of this attitude that pretends that the arithmetized version of calculus was the ineluctable goal of the development of calculus. In Robinson's (1966, p. 260) view, for instance, there is a stark contrast between the severity with which the ideas of infinitesimals are treated and "the leniency accorded to the lapses of the early proponents of the doctrine of limits."

My interest in this discussion is elsewhere. Following Luzin's foresight and other distinguished scholars and researchers, and reflecting on the lines of thought already discussed, I want to suggest a different strategy to teach this basic course, calculus, that continues to be a central organizing force in the curriculum. This goal is best served if we keep in mind a few basic conceptual differences between the intuitive approach to calculus and the formal approach known as *real analysis* for short. For instance, let me recall the ways in which Euler used infinitesimals for obtaining the power series corresponding to the exponential function. He made use of infinitesimals that obeyed the same rules of operations for finite numbers. If the only accepted numbers are the real numbers, then Euler's approach is not feasible. To obtain this result in the context of real analysis, one has to use notions such as the radius of convergence of a power series and Taylor expansions (Morgan 2005).

A key idea in Euler's approach is the use of the *local flatness* of a continuous function. It must be said that the domain of this continuous function is an infinitesimally enriched continuum. That is, if the function is continuous then an infinitesimal piece of its graph can be taken to be horizontal (with respect to the x-axis). This is what he did (previous section) with the graph of the function $y = 1/x$ between 1 and $1 + w$, w being an infinitesimal. His idea of continuous function was holistic, not point-wise as is taught currently. This latter conception of continuity led to results such as the existence of continuous non-differentiable functions that were instrumental for the arithmetization program that led to real analysis. Visualization lost its leading role and eventually its absence became a huge obstacle for the learner.

However, the idea of local flatness continues to be instrumental, for instance, in Tall's sensible approach to calculus (Tall 2013) as visual reasoning is a basic ingredient in the intuitive development of calculus. Visualization and movement are at the roots of what one could refer to as the calculus way of thinking. The arithmetization program, on the other hand, denies these ideas as constituents of a genuine mathematical way of thinking (Bottazzini 1986, pp. 91, 98). The graph of a differentiable function (a smooth curve) can be conceived of as a polygon with an infinite number of sides, each of infinitely small length. This conception, already present in L'Hôpital's book (Struik 1969, p. 314), has been creatively translated as the property of *local straightness* for a differentiable function in several of Tall's papers (see for instance Tall 2003, 2013).

Finally, to further highlight the differences in the approaches I have been discussing, let me bring to the fore the idea of limit and convergence. I will especially discuss, in Sect. 3, a very valuable work (Roh 2008) that besides its own results gathers up a substantial piece of the research done on this theme. Of particular interest is the cognitive dissonance between the intuitive, dynamic approach and the real analysis approach evidenced by Roh's research.

3 The pedagogical influences of a tradition

Student: The car has a speed of 50 miles an hour. What does that mean?

Teacher: Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|t_2 - t_1| < \delta$, then

$$\left|\frac{s_2 - s_1}{t_2 - t_1} - 50\right| < \varepsilon$$

Student: How in the world did anybody ever think of such an answer?

These are the first lines of Judith Grabiner's (1983) celebrated paper *Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus*. This hypothetical dialogue, between a teacher and a confused student, illustrates a drama that is repeated daily in classrooms. This drama is not new. In 1931, Mark Vygodskii published a book *Foundations of Infinitesimal Calculus* (see Demidov and Shenitzer 2000) and he meant, indeed, the word *infinitesimal* in the title. Vygodskii wrote that he intended to rescue the inductive way of thinking and the role of intuition. He criticized it as mistaken and harmful to present the fundamental concepts in a logical formal manner, as illustrated by the excerpts from Grabiner's paper. Vygodskii added:

> No matter how much individual authors try to simplify proofs, to avoid formal rigor, to introduce intuitive imagery and concrete problems, they invariably and above all, attempt to explain the formal scheme of modern analysis… This is the reason behind the depressing fact that the apparatus of analysis remains a dead apparatus in the hands of the students. (As cited in Demidov and Schenitzer 2000, p. 64)

It was not new in 1931, and it is not new today. Already installed in the twenty-first century, calculus continues to

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be mostly taught as if it were natural to introduce the mathematics of change and accumulation through a set of formal ideas and concepts, frozen like a block of ice. The internal connections of the idea of change are not correctly translated into this arithmetical approach of mathematical analysis. However, teachers feel guilty for abandoning the "right track," and books are written accordingly. This is something not fully acknowledged, though it must be said that in his foreword to Toeplitz's (2007) *The Calculus, A Genetic Approach*, the distinguished scholar David Bressoud wrote: "Though it would have been heresy to me earlier in my career, I have come to the conclusion that *most students of calculus are best served by avoiding any discussion of limits*" [emphasis added]. And he continued: "It is the students who have a good understanding of the methods and uses of calculus who are ready to learn about limits." This is sound advice indeed. Bressoud is clearly well aware of the rift between knowledge organized around intuitive experience and the logical and symbolic re-organization of this knowledge.

For instance, when introducing the integral in a first course, I have considered only the case of continuous functions and thus the integral is the artifact to compute the area that is already there, under the graph of the function. Later, in an analysis course, the area is *defined* by means of the integral. This kind of Copernican inversion clearly illustrates the difference between calculus and analysis. Analysis has its own goals and it does not seem prudent to try to introduce students to a complex of ideas through another discipline with a different structure and aiming at another direction. The precise logical definitions of limit, real number, continuity, derivative, integral, and so on, result from the refinements based on ideas and notions that are informal but intuitively clear. However, as the logical organization of analysis left its imprint on the curriculum, Luzin's voice was losing intensity. Indeed, the curricula based on these logical refinements ignored that what is first from the logical viewpoint is not necessarily first from the cognitive viewpoint.

In her comprehensive study, Kyeong Hah Roh (2008) explains that there is a paucity of studies that explore how the intuitive images of limit influence the formal definition. More precisely, her interest is to figure out why these intuitive images do not lead to the formal definitions. However, if one stays within a framework that takes the geometrical continuum (possibly an intuitive, informal, infinitesimally-enriched continuum) as a substrate, then the intuitive images probably will suffice to organize this body of introductory knowledge of calculus.

This is reminiscent of the approach followed by Cauchy, who was rather a systematizer whose goal had a quality quite similar to Euclid's in his *Elements of Geometry*. Cauchy's ideas have a geometric ground. His notion of continuity is derived from the holistic continuity of the straight line. When a function is continuous (an infinitesimal change in the independent variable produces an infinitesimal change in the value of the function) the graph of the function is a continuous curve. It is interesting to quote, as a sideline, from the introduction to his *Cours d'Analyse*: "As for the methods, I have sought to give them all the rigor which one demands from geometry, so that one need never rely on arguments drawn from the generality of algebra" (Bradley and Sandifer 2009, p. 1). On the other hand, if one follows the path traced originally by Bolzano, who suggested in his program that the formal approach (to calculus in particular) should expel motion from it, then one will have the situation described by Roh's work.

Roh's view is, apparently, that the formal definition takes priority over the intuitive images and that these intuitive images may become an obstacle to accessing the formal level. Even if the level of the population of students examined in this experiment validates the goal of understanding the formal definition, the work itself shows the existent rift between the intuitive images and the formal definitions. This is a valuable work indeed. Nevertheless I am using it to substantiate that her conclusions are in line, rather, with what can be naturally expected. Consequently, we should think perhaps of a different pedagogical strategy—or strategies. At some stages of development, certain previous conceptualizations must be transformed if we want to access a new and possibly higher level of understanding. Nevertheless, this new level is not unique: It depends on the results we are trying to achieve. In this respect Pontryagin wrote:

> In a historical sense integral and differential calculus had already been among the established areas of mathematics long before the theory of limits. The latter originated as superstructure over an existent theory. Many physicists opine that the so-called rigorous definitions are, in no way, necessary for the satisfactory comprehension of differential and integral calculus. I share this opinion. (As cited in Gordon et al. 2002, p. 7)

The formal structures are not Kantian, inborn structures. It is the activity of human beings that generated these structures and superstructures from human experience. Roh's (2008) work is well documented and in this sense it can be taken as a synthesis. She mentions that also other researchers found that most students do not adopt the formal definition of limit (p. 218). In this text I have provided ample evidence that indicates that, in the experience of many professionals, it is not sensible to introduce a formal definition that clearly contradicts the intuition of students who will feel the introduction of the formal definition as something disconnected from their experiences. Moving

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further in Roh's text, one can read that "the words used in expressing the dynamic imagery of limits, such as 'approaching,' or 'getting close to,' do not precisely convey the mathematical meaning of the concept of limit" (p. 218).

This is precisely Luzin's claim whilst contrasting the formal structure with basic intuitions, saying: *whether this* [the formal structure] *corresponds to what is in the depths of our consciousness is a very different question.* Is the mathematical meaning of the concept totally captured by the formal definition? A positive answer would imply that one could separate meaning from human experience. It is opportune here to notice that we are dealing with students' cognition, not simply established knowledge. Indeed, it is clear at this time that students have problems understanding formal concepts. We cannot change the students but we can change our teaching strategies. Perhaps those misunderstandings we attribute to the students mean, indeed, that we misunderstand the students as cognitive beings.

The students, when instructed to reflect and accommodate themselves to the complex symbolic structures of formal definitions, find that the intuitive images become obstacles. This is Roh's (2008) finding. But one could try to see the problem from the side of the intuitive images that, cognitively, takes precedence in elementary calculus. What is the kind of formalization that corresponds to those intuitive images? Teachers know that formalization never fits exactly with the intuitive idea, with the insight that gave rise to that formalization.

As people represent their experience in diverse symbolic representational systems, those experiences gain in objectivity. These symbolic representations can be explored and they cast light on the original intuitions and insights. This is the case with the study of change and accumulation by means of analytic representations. One can begin a systematic exploration through the study of *the tangent to the graph of the function*. Here one has an objective way to represent symbolically the phenomenon of variation and eventually it will pay back, providing a more systematic version of change. Increasingly, experiences and the symbolizations of those experiences become more and more cognitively equivalent.

Donald (2001) writes: "Symbols can mediate certain forms of thought and make it possible to formulate new kinds of concepts provided that the underlying capacity for understanding is already there" (p. 121). This observation alerts the teacher to the role of context. The interactions between intuitions and symbolizations deepen as the students' knowledge grows. This theme demands a permanent call to prudence. Human culture is profoundly symbolic. It is natural, then, to fail to distinguish between the mode of existence of our symbolic universes and our daily realities.

4 First steps towards a possible answer

On the occasion of his plenary lecture *Modern Mathematics: Does It Exist?* René Thom (1973) expressed that "the real problem, which confronts mathematics teaching, is not that of rigor, but the problem of the development of 'meaning,' of the 'existence' of mathematical objects" (p. 202). The problem is, indeed, very delicate because mathematical objects are not, obviously, material but conceptual objects and this implies that there is no direct access to these objects. The only possibility is to access the object through a symbolic representation. As Otte (2006) has written, "a mathematical object, such as a 'number' or 'function' does not exist independently of the totality of its possible representations," and he continues to explain that the object "must not be confused with any particular representation, either" (p. 17). Each representation provides a door of access, but what one "finds" through it must not be confused with the object under consideration.

Once a new representation is added or considered, the previous object is then transformed. There is a pressing educational need to look for a new representation: the need to help students understand an idea, a concept. Consequently, finding a new representation might have an epistemological and/or an educational gain. As educators, though, we need to be prudent to avoid saturating students with too many representations.

The system that Leibniz invented to represent his calculus ideas was a sensible medium that would "guide the mind as do the lines drawn in geometry." Indeed, his system played an instrumental role in the development of calculus.

In a static medium we can display information and knowledge by means of graphical representations. This is what Leibniz did in an outstanding way. On the other hand, in a digital-dynamic medium, those graphical representations become much more flexible and one can "bend" and/or "stretch" them and draw the tangent line and drag it along the graph of a given function, for instance. These intentional actions in a digital-dynamic medium generate a feeling of objectivity associated, in the mind of the user, that is, of the student/teacher, with the existence of the mathematical object under consideration. This objectivity gives the impression of definiteness of the object and as such is taught in schools. But this object is always growing, penetrating other mathematical realms, and consequently establishing new and sometimes unexpected connections.

The construction of the tangent line (or an approximation to the area under the graph of a function) incorporates motion that the user can control and explore. Thus, the user can see, emerging from the screen, an evolving representation that explains visually the concept that the user tries to understand. The screen becomes a kind of virtual reality

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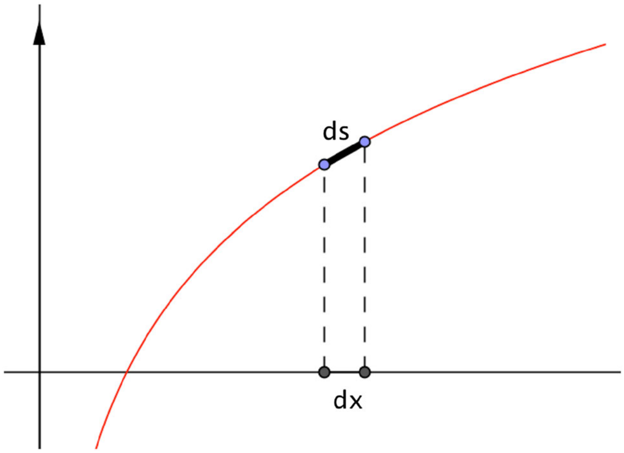

Fig. 8 Infinitesimal side

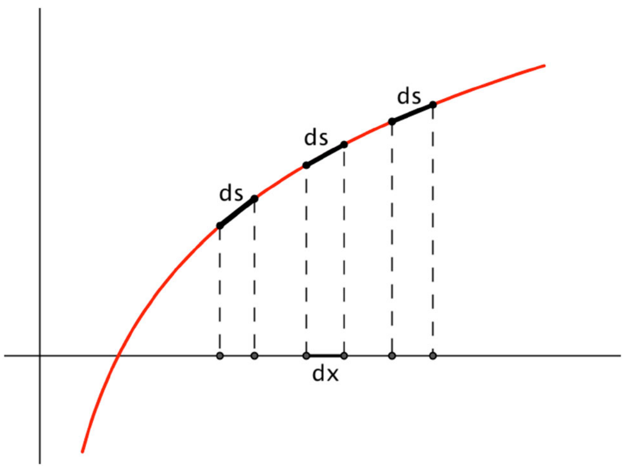

Fig. 9 Curve as polygon

where the user can explore and develop new mathematical experiences because of the mathematical expressivity that the digital medium makes possible. In a certain sense, motion becomes part of the representation itself and translates well the motion belonging to earlier calculus concepts. This translation is key for finding ways to blend the intuitive ideas with the digital-dynamic embodiment of these ideas.

Now, I offer some glimpses of a digital translation of basic ideas of calculus. The zoom-in facility that comes with GeoGebra makes this software adequate for our goals. The digital version of the L'Hôpital axiom (local straightness of a smooth curve) is represented in Fig. 8.

One can slide the side *ds* and see that it dynamically *describes* the curve. The differential *dx* is key in this process because it controls the size of *ds*. If one activates the *trace* of the dynamic medium then, whilst *ds* is sliding, one can appreciate how its trace covers the curve (Fig. 9).

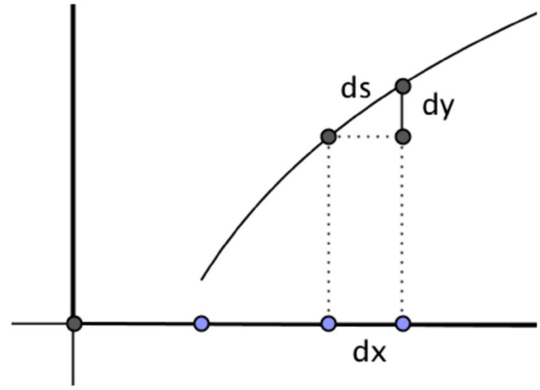

Fig. 10 Slope of curve

This is a visual metaphor that helps the students understand how the smooth curve is composed of the infinitesimal segments *ds*. It must be said, though, that this metaphor is feasible because seeing the polygonal line as a smooth curve depends on the pixel size and the resolution capability of the human eye.

Unfortunately, the written narrative is insufficient to translate the force of the dynamic unfolding of the whole situation. In fact, motion is a dimension of the (dynamic) figure, and here, on the printed page, it is absent.

Once one has the figure, it is almost immediate to complete the Leibniz triangle to measure the slope along the curve, as is shown in Fig. 10.

Having calculated the slope at a point, it is possible, sliding the point in the domain of the function, to draw the graph of the derivative. I am using the graph of the logarithmic function to illustrate these ideas, so in Fig. 11 we see the graph of the hyperbola, $y = 1/x$, that is, the derivative of $\log(x)$.

The point Q is key to this construction. The segment FG (in the figure) controls the length of *dx*. It must be observed that the shorter FG, the more precise the curve that represents the derivate function. Here one is taking this notion of derivative back to its basic intuitions. Unfortunately as I have said before, the dynamics of the figure is absent from the printed page. This is the local organization to introduce and reflect on the derivative of a function. We have called it *taking the slope* and it casts light on the kernel of the fundamental theorem of calculus.

I have just highlighted some examples, but it is possible to go further in this direction, for instance discussing integration. As is the case with any mediating artifact, the artifact is not neutral from an epistemic point of view. This is, in my view, the profound idea embodied in the concept *humans-with-media* (Borba and Villarreal 2005). Knowledge is not independent from the artifacts that *mediate* the appropriation of knowledge, in particular, mathematical knowledge: human cognition is mediated cognition.

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We all are witnessing the migration of mathematical objects to computer screens and other digital media. In the case of calculus, I have already mentioned that those representations embody motion as an inherent dimension of the objects. The mutual transformation of user and medium goes beyond the traditional interactions between a user and a rigid tool.

Calculus is an inherited body of knowledge. During its production pre-digital and digital systems of representations have been employed. Each one of them impacts the nature of the knowledge produced. An object that can be represented in both static and dynamic media acquires a hybrid nature: the digital embodiment of that object enables it to "speak" properties that previously remained silent, implicit, or simply hidden.

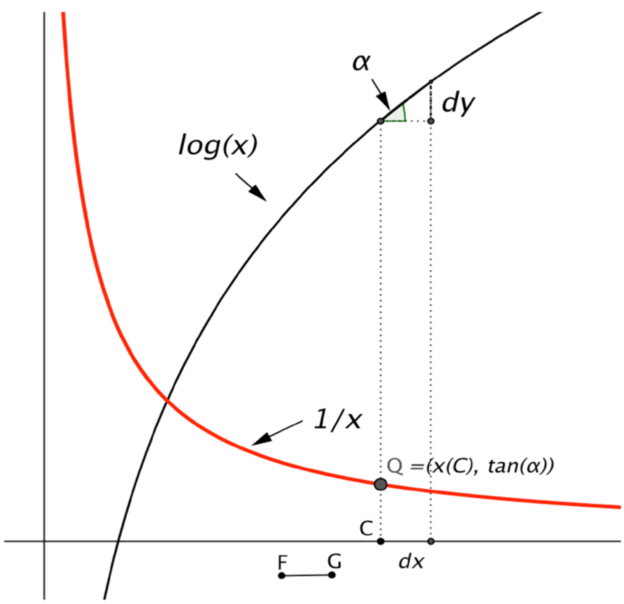


Fig. 11 Graph of derivate function

To complement our last exemplar, I indicate how to recover the logarithmic function from its derivative. Let us begin with the area under the graph of the function $y = 1/x$ and see how to produce the logarithm as its integral.

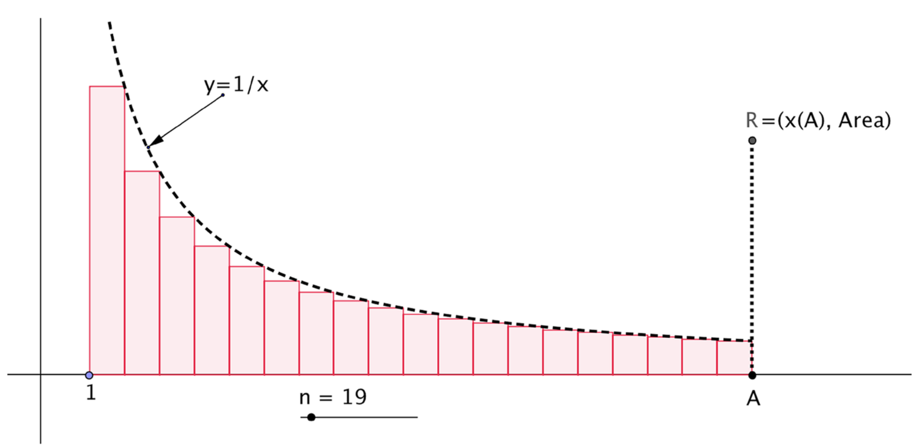
The point R (in Fig. 12) has coordinates x(A), the x-coordinate of A, and, as y-coordinate, the accumulated area from 1 to A. If one increases the number of rectangles then the upper side of the rectangles (in this lower-sum) will approach the graph of $y = 1/x$.

The slider ($n = 19$) indicated in Fig. 12 will play the role of "enhancer of the approximation" to the area as well as to the function $1/x$ itself by means of the sequence of step functions (the upper sides of the rectangles). The next figure will show the "integral" curve as n is increased ($n = 79$ in Fig. 13):

Let me emphasize that digital media enhance substantially the expressive capacity of both teachers and students. There are new phenomena on the screen they can refer to and this offers opportunities to connect with previous knowledge. For instance, when the user is increasing the numerical value of the slider (Fig. 13) s/he can see how her/his action is affecting the approximation process. It is rewarding for the students to work with this construction. Dragging the point A to the left and right whilst the number of rectangles (n) is fixed is an action of the student that corresponds to the drawing of an approximation to the curve log(x). This is important: the medium reacts to the user's actions—I will come back to this point later. Eventually it becomes clear that if, previously, s/he increases the number of rectangles through the slider, then the curve that is drawn whilst dragging A will be a better approximation to the graph of log(x).

What is new here is the perceived dynamic unfolding of the figures, visually corresponding the analytic and numerical phenomenon of approximation. But this action is shared with the medium. In fact, the user *co-acts* with the medium (Moreno-Armella and Hegedus 2009, p. 510), that

Fig. 12 Hyperbolic Area



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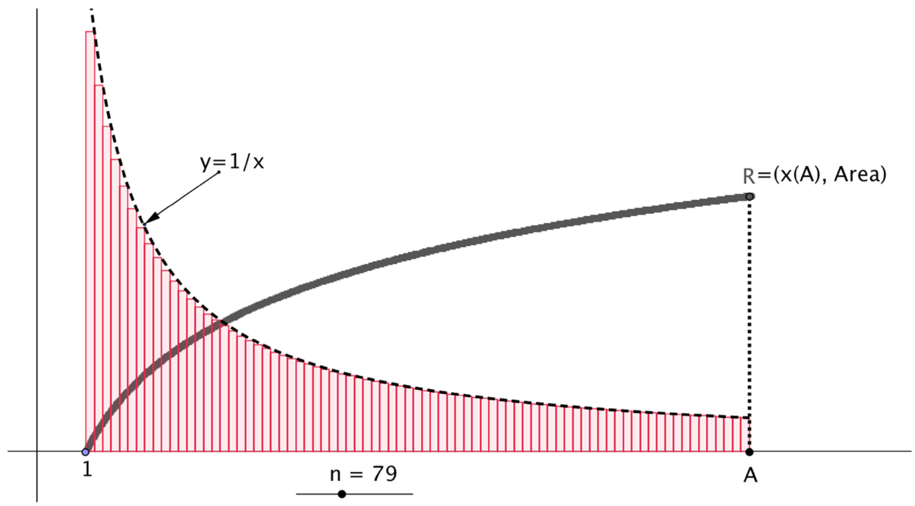

Fig. 13 Integrating the derivative

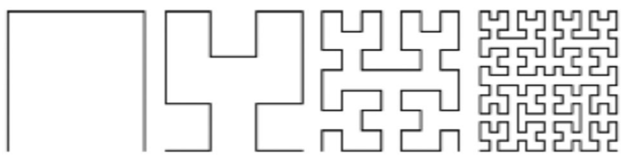

Fig. 14 Hilbert curve

is, the user and the medium react to each other iteratively. If one thinks of the classroom, the medium should be understood as including students, teachers, and the technology available. In such a setting one can learn from the others, through the others, and with the others. Notice that all these possibilities may be present while co-acting with a technology.

The digital and executable representational systems have induced a new mathematical realism. As we have written elsewhere, mathematical representations co-evolve with their mathematical referents and the induced semiotic objectivity makes it possible for them to be taken as shared in a community of practice (Moreno-Armella and Sriraman 2010, p. 220).

In calculus, in particular, one needs to devise transition strategies to deal, accordingly, with the new representational systems and the hybrid nature of mathematical objects. Consider for instance the Hilbert space-filling curve (http://en.wikipedia.org/wiki/File:Hilbert_curve.gif) explained by Hilbert in 1891 (Fig. 14). In the digital rendering, the curve recovers its dynamic nature (Moreno-Armella and Hegedus 2013, p. 36).

The original proof is beyond our students'/teachers' scope. However, when one takes advantage of its dynamic representation, the theorem migrates to the digital medium in the form:

> Given a screen resolution, there is always one level in the iterative production of the curve that fills that screen.

This calls attention to the epistemic fact that the proof is relative to the medium in which it takes place and relative as well to the mathematical culture of the time. Euler's proofs, for instance, could not be published today. They belong to a different mathematical culture. Hilbert's digital theorem fits well with the exploration of the visual phenomenology of dynamic objects on the screen. This kind of exploration begins to develop in the student/teacher a cognitive tool for re-conceptualizing the hybrid mathematical objects. The Leibniz triangle, for instance, is re-conceptualized as the generator of a graph, in fact, the corresponding derivative graph. The local organization designed from the digital Leibniz triangle enables the user to see through the executable representations involved, rendering visible the underlying mathematical structure.

Through this lived experience of the students/teachers they have been able to witness how the accessibility of mathematical ideas and their nature are transformed by the symbol-mediated experience made possible by the digital embodiment of motion. The issue is not whether to use dynamic images in teaching, but rather how to induce dynamic images that cohere with basic intuitions such as the ones leading to Leibniz's or Euler's results. When this is possible, one can develop a more natural organization of calculus in schools. Roh's (2008) findings have shown that the usual dynamic images are not compatible with the formal definitions of mathematical analysis. Taking into account the accumulated experience and the empirical work advanced so far, it seems that it is extremely complex at least—from the educational point of view—to make the intuitive and the formal compatible at the level of basic

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calculus. Nevertheless, a line of feasible development can be further advanced by producing a narrative of calculus based on basic ideas such as the geometrical continuum, the intermediate value theorem in this dynamic version, and a dynamic version of convergence: *approaching indefinitely*.

Of course, one cannot take as equivalent the examples from history and the examples from today's classroom. Intuition is subtly modified by the presence of cultural symbolic environments but, at the same time, intuition is always there, playing a main role in the process of developing knowledge. This can be illustrated with L'Hôpital's axiom—a curve can be conceived of as a polygon whose sides are infinitesimally small. A dynamic medium enables the student to see and understand a digital embodiment of this axiom. In fact, the infinitesimal and the digital models are both *discrete* models. In the digital model the possibility of zooming in on a graph corresponds (intuitively) to taking infinitesimals. It is important to make this explicit: I am elaborating this equivalence at the cognitive level, not the mathematical level. The digital medium offers a model of the infinitesimal side of a polygon corresponding to a smooth curve, that is, the graph of a differentiable function.

The dilemma tradition/innovation can be considered by means of digital representations of mathematical entities. Doing so reveals properties of these entities that lie hidden or opaque, to begin with. The goal, let us insist, is to develop a *transitional* way of thinking more in agreement with the requirements of education today. At this point, it is important that these reflections take place in a classroom with teachers.

To advance these reflections it has been important to translate basic ideas from the infinitesimal into the digital model. Metaphors have been the artifacts to make this feasible and, as suggested previously when Fig. 9 was discussed, seeing the graph on the screen as a smooth graph is an iconic interpretation of the graph. An iconic representation, it is well known, is one whose form directly reflects the thing it signifies. During the work with the students/teachers, whilst the digital medium was introduced, it took a serious effort to make it clear for them that the corresponding digital representations were not iconic, but metaphorical. The meaning of metaphor is "to transfer." The traditional, written representations were transferred from the paper onto the screen and there, in the new medium, they became executable. Hence, being able to drag a figure, to modify at will the shape of a graph, are actions that awaken the strong feeling that one is working directly with the real mathematical entity. In consequence, the students/teachers were frequently persuaded of the iconicity of these digital representations. Discussing the size of the pixel, to say it so, was instrumental in banishing, in general, this misconception about the iconic character of mathematical representations.

All these efforts to teach calculus are in need of increasing levels of understanding of the mathematics involved, and the present and future roles of digital representations. More precisely: it is necessary to bridge the fissure between the intuitive ideas of calculus, as presented succinctly in the former sections of this paper, and the digital embodiments of those same ideas, involving motion and variation, as discussed in this section through executable representations. This is a goal that I cannot imagine if the education of the teachers is left aside.

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